Independence Results at the Successors of Singular Cardinals

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Cardinal Arithmetic and Independence
Singular Cardinals
Forcing dos and don'ts
Iterations and successors of singulars

Ordinals, Singular and regular cardinals

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This theorem introduced a method, *forcing*, which can be used to extend models of ZFC to bigger models of ZFC, by adding to the first model a *filter* in a certain partial order and closing off under *evaluations of names*.

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Some independence

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Theorem

(Magidor) Assuming the consistency of rather large cardinals, it is consistent that $\aleph_\omega$ be the first cardinal to fail GCH.

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So we work with the partial order of finite functions from $\omega \to 2$, ordered by inclusion (call this Add$(\omega, 2)$).

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2. Suppose that ZFC could prove CH, then the proof would involve a finite fragment Γ of ZFC, fix such a Γ.

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Using similar density arguments as in the case of Cohen forcing, we can see that Prikry forcing adds an ω-sequence cofinal in κ. The point of the measure is to show that the cardinals are preserved.

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Many of these developments are described in Gitik's article in the Handbook of set theory.

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1. move the emphasis from the power set problem to include more combinatorial problems and
2. develop forcing techniques that will allow us to deal with the successors of singular cardinals.

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The method is to do an iteration on κ while it is still supercompact and build a special ultrafilter that will be used for singularising at the end in a way that once the cardinal is singularised the results of the iteration still hold.

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Which cardinals have or can have the tree property? This is a complicated question which is still not entirely understood. Even for $\aleph_2$ alone to have the tree property one needs a large cardinal (Mitchell and Silver).

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(Neeman) From large cardinals, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that the Singular Cardinal Hypothesis fails at κ and the tree property holds at κ^+.

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The work on the tree property was continued and generalised in many directions in a series of papers by Dima Sinapova, who obtained other cofinalities, several cardinalas at once etc.