

# TOPOLOGICAL MODELLING OF KNOWLEDGE CHANGE

## LECTURE 2

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PhDs in Logic VIII  
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# OUTLINE

## INTRODUCTION

Learning and Solving in the Limit  
Topological Toolbox

## CHARACTERIZATION OF LEARNABILITY AND SOLVABILITY

## CONSTRUCTIVE ORDER-DRIVEN LEARNING

## TOWARDS EPISTEMIC LOGIC OF LEARNABILITY

Topological semantics  
Doxastic interpretations

## INTERMEDIATE (BUT INTERESTING ) CONCLUSIONS

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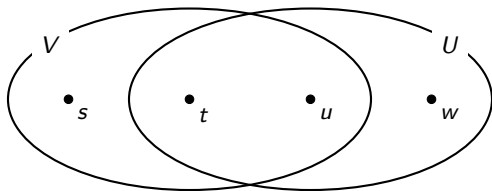
# MARRYING TOPOLOGY AND EPISTEMOLOGY (VIA LEARNABILITY)

1. Learnability and solvability in terms of topological separation properties.
2. Constructive, order-based learning by updating.
3. Consequences for logic of belief (doxastic logic).

# EPISTEMIC SPACES AND OBSERVABLES

## DEFINITION

An *epistemic space* is a pair  $\mathbb{S} = (S, \mathcal{O})$  consisting of a state space (a set of possible worlds)  $S$  and a countable set of observable properties  $\mathcal{O} \subseteq \mathcal{P}(S)$ .



# LEARNING: STREAMS OF OBSERVABLES

## DEFINITION

Let  $\mathbb{S} = (S, \mathcal{O})$  be an epistemic space.

- ▶ A *data stream* is an infinite sequence  $\vec{O} = (O_0, O_1, \dots)$  of data from  $\mathcal{O}$ .
- ▶ A *data sequence* is a finite sequence  $\sigma = (\sigma_0, \dots, \sigma_n)$ .

## DEFINITION

Take  $\mathbb{S} = (S, \mathcal{O})$  and  $s \in S$ . A data stream  $\vec{O}$  is:

- ▶ *sound with respect to  $s$*  iff every element listed in  $\vec{O}$  is true in  $s$ .
- ▶ *complete with respect to  $s$*  iff every observable true in  $s$  is listed in  $\vec{O}$ .

We assume that data streams are sound and complete.

# LEARNING: LEARNERS AND CONJECTURES

## DEFINITION

Let  $\mathbb{S} = (S, \mathcal{O})$  be an epistemic space and let  $\sigma_0, \dots, \sigma_n \in \mathcal{O}$ . A *learner* is a function  $L$  that on the input of  $\mathbb{S}$  and data sequence  $(\sigma_0, \dots, \sigma_n)$  outputs some set of worlds  $L(\mathbb{S}, (\sigma_0, \dots, \sigma_n)) \subseteq S$ , called a *conjecture*.

## DEFINITION

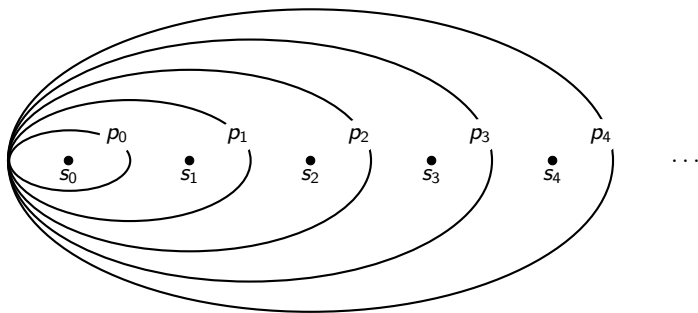
$\mathbb{S} = (S, \mathcal{O})$  is *learnable by  $L$*  if for every state  $s \in S$  we have that for every sound and complete data stream  $\vec{O}$  for  $s$ , there is  $n \in \mathbb{N}$  s.t.:

$$L(\mathbb{S}, (O_0, \dots, O_k)) = \{s\} \text{ for all } k \geq n.$$

An epistemic space  $\mathbb{S}$  is *learnable* if it is learnable by a learner  $L$ .

## EXAMPLE OF A LEARNABLE SPACE

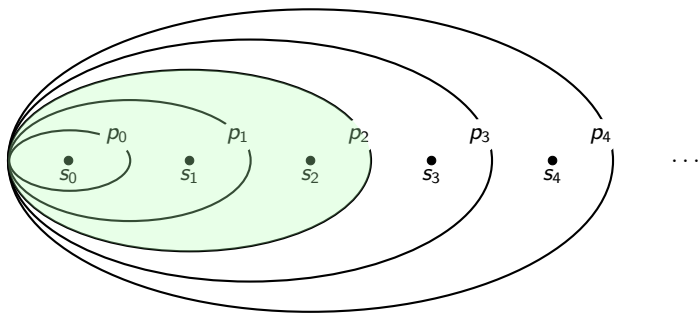
Let  $\mathbb{S} = (S, \mathcal{O})$  such that  $S = \{s_n \mid n \in \mathbb{N}\}$ ,  $\mathcal{O} = \{p_i \mid i \in \mathbb{N}\}$ , and for any  $k \in \mathbb{N}$ ,  $p_k = \{s_i \mid 0 \leq i \leq k\}$ .  $\mathbb{S}$  is learnable.





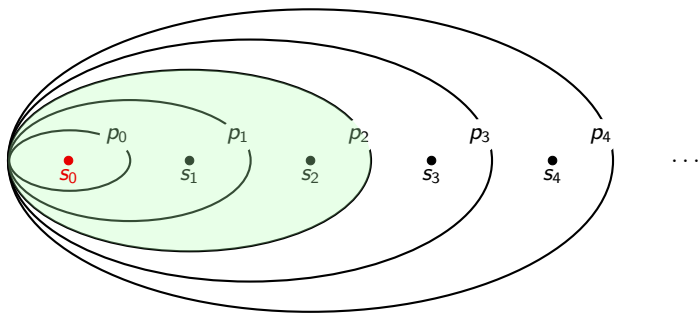
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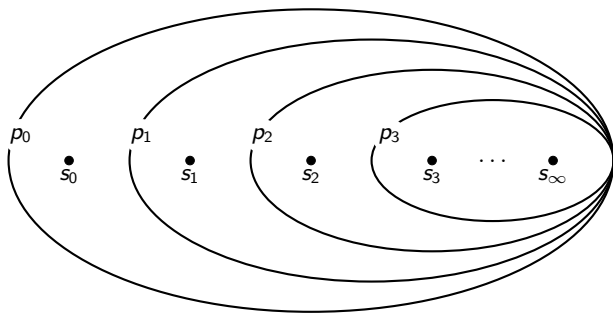
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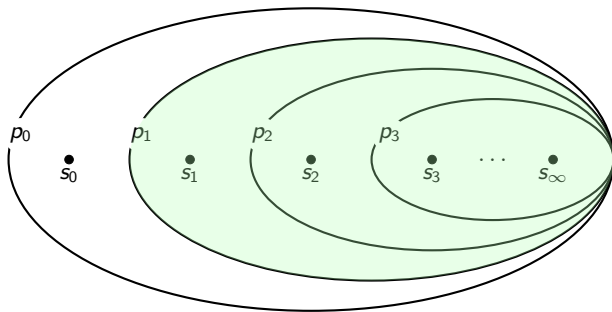
## EXAMPLE OF A NON-LEARNABLE SPACE

Consider  $\mathbb{S} = (S, \mathcal{O})$ , where  $S := \{s_n \mid n \in \mathbb{N}\} \cup \{s_\infty\}$ , and  $\mathcal{O} = \{p_i \mid i \in \mathbb{N}\}$ , and for any  $k \in \mathbb{N}$ ,  $p_k := \{s_k, s_{k+1}, \dots\} \cup \{s_\infty\}$ .  $\mathbb{S}$  is not learnable.



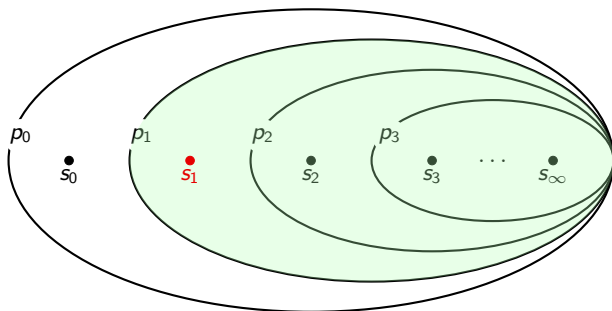
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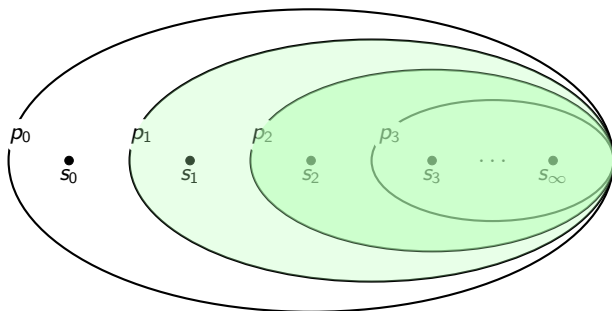
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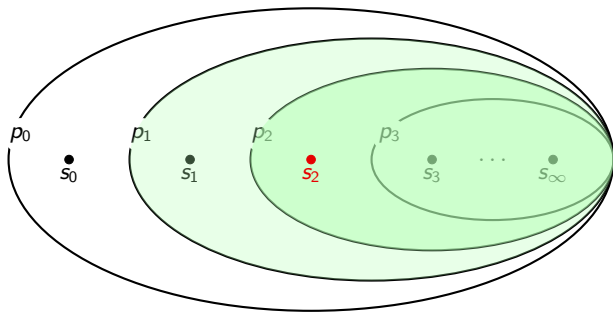
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# QUESTIONS, ANSWERS, AND PROBLEMS

## DEFINITION

A *question*  $Q$  is a partition of  $S$ , whose cells  $A_i$  are called *answers to*  $Q$ .  
Given  $s \in A \subseteq S$ ,  $A \in Q$  is called *the answer to*  $Q$  at  $s$ , denoted  $A_s$ .

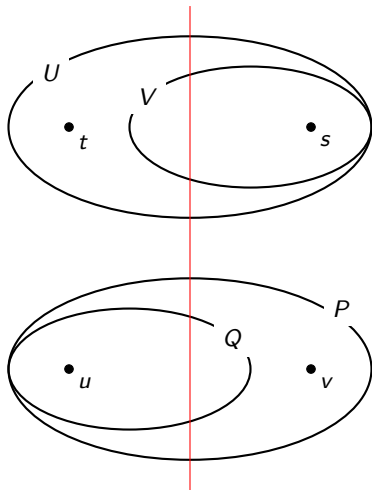
## DEFINITION

$Q'$  is a *refinement* of  $Q$  if all answers of  $Q$  is a disjoint union of answers of  $Q'$ .

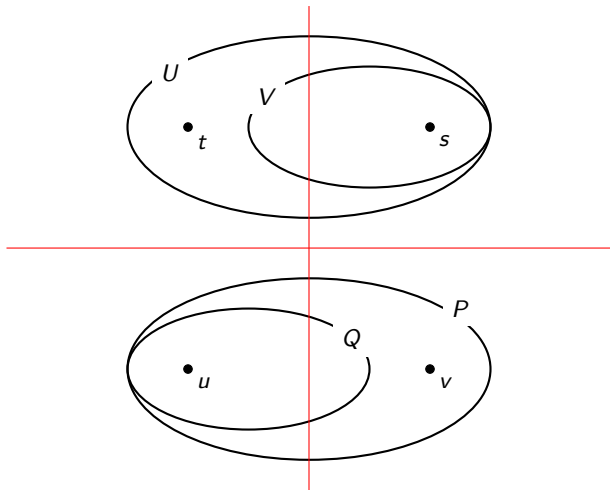
## DEFINITION

A *problem*  $\mathbb{P}$  is a pair  $(\mathbb{S}, Q)$  consisting of  $\mathbb{S} = (S, \mathcal{O})$  and  $Q$  over  $S$ .  
 $\mathbb{P}' = (\mathbb{S}, Q')$  is a *refinement* of  $\mathbb{P} = (\mathbb{S}, Q)$  if  $Q'$  is a refinement of  $Q$ .

# ILLUSTRATION



# ILLUSTRATION



# SOLVING IN THE LIMIT

## DEFINITION

A learning method  $L$  solves a problem  $\mathbb{P} = (\mathbb{S}, \mathcal{Q})$  in the limit iff for every state  $s \in S$  and every data stream  $\vec{O}$  for  $s$ , there exists some  $k \in \mathbb{N}$  such that:

$$L(\mathbb{S}, \vec{O}[n]) \subseteq A_s \text{ for all } n \geq k.$$

A problem is *solvable in the limit* if there is a learner that solves it in the limit.

# GENERAL TOPOLOGY

## DEFINITION

A topology  $\tau$  over a set  $S$  is a collection of subsets of  $S$  (open sets) s.t.:

1.  $\emptyset \in \tau$ ,
2.  $S \in \tau$ ,
3. for any  $X \subseteq \tau$ ,  $\bigcup X \in \tau$ , and
4. for any finite  $X \subseteq \tau$  we have  $\bigcap X \in \tau$ .

## DEFINITION

Take a set  $X \subseteq S$ .

1. The *interior* of  $X$ :  $Int(X) = \bigcup\{U \in \tau \mid U \subseteq X\}$ .
2. A subset  $Y \subseteq S$  is *closed* if and only if its complement,  $Y^c$  is open.
3. The *closure* of  $X$ :  $\bar{X} = (Int(X^c))^c = \bigcap\{Y \mid X \subseteq Y \text{ and } Y \text{ is closed}\}$ .

# SEPARABILITY OF STATES

## DEFINITION

A topology  $\tau$  over  $S$  is  $T1$  (strongly separated) just if every state  $s \in S$  is separable from every other state in  $S$ , i.e., for all  $s, t \in S$ , if  $s \neq t$  then there is an  $U \in \tau$  such that  $s \in U$  and  $t \notin U$ .

## DEFINITION

A topology  $\tau$  over  $S$  is  $T0$  (weakly separated) just if all distinct states are separable one way or another, i.e., for all  $s, t \in S$  if  $s \neq t$  then there is and  $U \in \tau$  such that either  $s \in U, t \notin U$  or  $s \notin U, t \in U$ .

# ILLUSTRATION

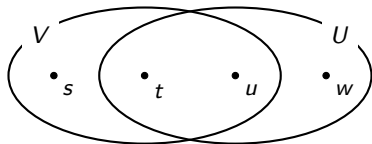


FIGURE:  $t$  and  $u$  are not separable

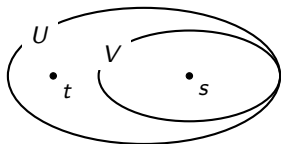


FIGURE: weakly separated space,  $T_0$

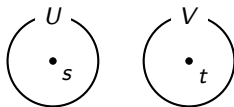


FIGURE: strongly separated space,  $T_1$

# LOCALLY CLOSED AND CONSTRUCTIBLE SETS

## DEFINITION

A set  $A$  is *locally closed* if  $A = U \cap C$ , where  $U$  is open and  $C$  is closed.

A set is *constructible* if it is a finite disjoint union of locally closed sets.

## DEFINITION

A topology  $\tau$  is  $T_d$  iff for every  $s \in S$  there is a  $U \in \tau$  such that  $U \setminus \{s\} \in \tau$ , i.e., for every  $s \in S$  there is a  $U \in \tau$  such that  $\{s\} = U \cap \overline{\{s\}}$ .

$T_d$  is a separation property between  $T_0$  and  $T_1$ .



# $\omega$ -CONSTRUCTIBLE SETS

## DEFINITION

An  $\omega$ -constructible set is a countable union of locally closed sets.

## PROPOSITION

Every  $\omega$ -constructible set is a disjoint countable union of locally closed sets.

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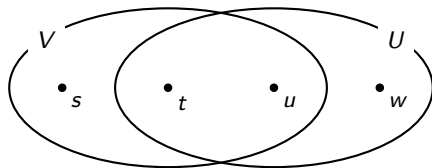
## INTERMEDIATE (BUT INTERESTING ) CONCLUSIONS

# THE TOPOLOGY ASSOCIATED WITH AN EPISTEMIC SPACE

## DEFINITION

The topology  $\tau_{\mathbb{S}}$  associated with an epistemic space  $\mathbb{S} = (S, \mathcal{O})$  is a collection of subsets of  $S$  of the following properties:

1. for any  $O \in \mathcal{O}$  it is the case that  $O \in \tau_{\mathbb{S}}$
2.  $\emptyset \in \tau_{\mathbb{S}}$ ,
3.  $S \in \tau_{\mathbb{S}}$ ,
4. for any  $U \subseteq \tau_{\mathbb{S}}$ ,  $\bigcup U \in \tau_{\mathbb{S}}$ , and
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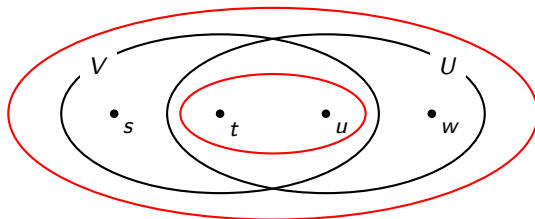


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5. for any  $x, y \in \tau_{\mathbb{S}}$  we have  $x \cap y \in \tau_{\mathbb{S}}$ .



# CHARACTERIZATION OF SOLVABILITY IN THE LIMIT

## THEOREM

*A problem  $\mathbb{P} = (\mathbb{S}, \mathcal{Q})$  is solvable in the limit iff  $\mathcal{Q}$  has a countable locally closed refinement.*

## COROLLARY

*An epistemic space  $\mathbb{S} = (S, \mathcal{O})$  is learnable in the limit iff it is countable and satisfies the  $T_d$  separation axiom.*

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# ORDER-DRIVEN LEARNING: MOTIVATION

- ▶ Belief Revision: minimal states give beliefs.
- ▶ Computational Learning Theory: co-learning, learning by erasing.
- ▶ Philosophy of Science: Ockham's razor.

# CONDITIONING

## DEFINITION

Conditioning wrt a prior  $\leq$  on  $S$ , is defined in the following way:

$$L_{\leq}(O_1, \dots, O_n) := \text{Min}_{\leq} \left( \bigcap_{i=1}^n O_i \right)$$

whenever  $\bigcap_i O_i$  has any minimal elements; and otherwise:

$$L_{\leq}(O_1, \dots, O_n) := \bigcap_{i=1}^n O_i.$$

## DEFINITION

Conditioning is said to be *standard* if the prior  $\leq$  is *well-founded*.



# DIRECT SOLVABILITY BY CONDITIONING

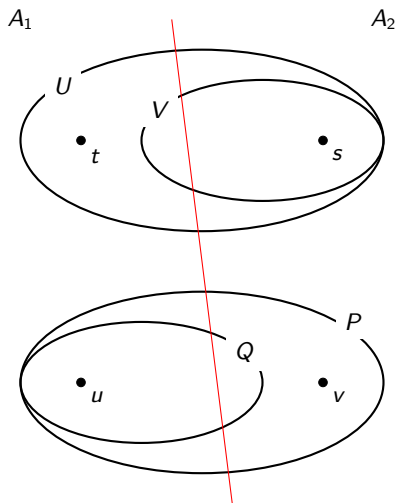
## DEFINITION

Given a question  $Q$  on an epistemic space  $\mathbb{S} = (S, \mathcal{O})$ , any total order  $\trianglelefteq \subseteq Q \times Q$  on  $Q$  induces in a canonical way a total preorder  $\leq \subseteq S \times S$ :

$$s \leq t \text{ iff } A_s \trianglelefteq A_t.$$

A problem  $\mathbb{P} = (\mathbb{S}, Q)$  is *directly solvable by conditioning* if it is solvable by conditioning with respect to a total order  $\trianglelefteq \subseteq Q \times Q$ .

# ORDERING THE STATES V. ORDERING THE ANSWERS



# LINEAR SEPARATION

## DEFINITION

A partition  $\mathcal{Q}$  of a topological space  $(S, \tau)$  is *linearly separated* if there exists some *total order*  $\triangleleft$  on  $\mathcal{Q}$  and a map  $O : \mathcal{Q} \rightarrow \tau$ , s.t. for all cells  $A, B \in \mathcal{Q}$ :

1.  $A \subseteq O_A$ ;
2. if  $B \triangleleft A$  then  $O_A \cap B = \emptyset$  (where  $\triangleleft$  is the corresponding *strict order*).

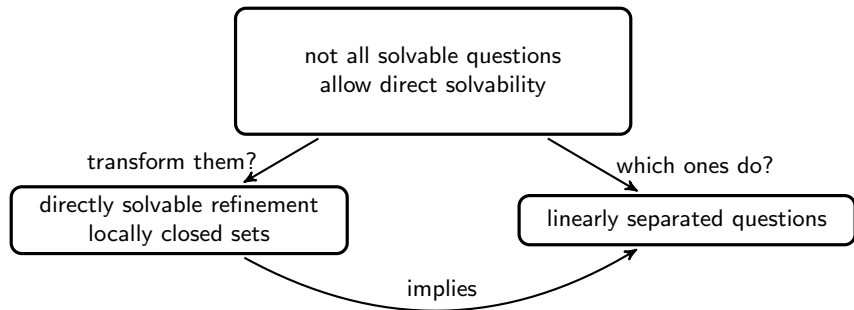
## THEOREM

$\mathbb{P} = (\mathbb{S}, \mathcal{Q})$  is *directly solvable by conditioning* iff  $\mathcal{Q}$  is *linearly separated*.

# UNIVERSALITY: BUILDING STRONGER REFINEMENTS

## THEOREM

*Conditioning is a universal problem solving method.*



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## INTERMEDIATE (BUT INTERESTING ) CONCLUSIONS

# LOGIC FOR LEARNABILITY

Since learnability is about potentially successful changes of **beliefs** one expects some doxastic logic to capture it and to reason about it.

# RELATIONAL SEMANTICS FOR MODAL LOGIC

## DEFINITION (SYNTAX)

Take countable set of propositional symbols  $P$ .

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi,$$

for all  $p \in P$ , the usual abbreviations are  $\vee$ ,  $\rightarrow$ , and  $\Diamond$ .

# RELATIONAL SEMANTICS FOR MODAL LOGIC

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## DEFINITION (SEMANTICS)

Given a model  $M = (W, R, v)$ , where  $v : P \rightarrow \wp(W)$ , and a state  $x \in W$ :

$M, x \models p$	iff	$x \in v(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	for all $y \in W$ : if $xRy$ then $M, y \models \varphi$
and dually:		
$M, x \models \Diamond\varphi$	iff	there is $y \in W$ : $xRy$ and $M, y \models \varphi$



# SOME AXIOMS AND THEIR EPISTEMIC MEANING

## Rules

(MP) if  $\vdash \varphi$  and  $\vdash \varphi \rightarrow \psi$ , then  $\vdash \psi$

(N) if  $\vdash \varphi$ , then  $\vdash \Box\varphi$

## Axioms

(K)  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$  (omniscience)

(T)  $\Box\varphi \rightarrow \varphi$  (truthfulness/reflexivity)

(D)  $\Box\varphi \rightarrow \neg\Box\neg\varphi$  (consistency/seriality)

(4)  $\Box\varphi \rightarrow \Box\Box\varphi$  (positive introspection/transitivity)

(5)  $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$  (negative introspection/Euclidean-ness)

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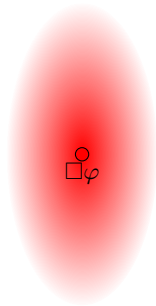
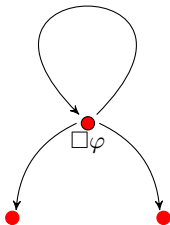
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$Ax$  is a logic of a class of models  $\mathcal{M}$  iff  $Ax$  is sound and complete wrt  $\mathcal{M}$ .

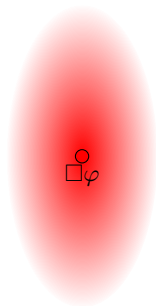
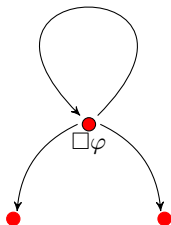
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RELATIONAL  $\Box$  VS TOPOLOGICAL  $\Box := \text{Int}$



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RELATIONAL  $\Box$  VS TOPOLOGICAL  $\Box := \text{Int}$



## DEFINITION

Let  $P$  be a set of propositional symbols. A topological model (or a topo-model)  $M = (X, \mathcal{O}, \nu)$  is a topological space  $\tau = (X, \mathcal{O})$  together with a valuation function  $\nu : P \rightarrow \wp(X)$ .

# TOPOLOGICAL TOPO-SEMANTICS FOR MODAL LOGIC

## DEFINITION

Truth of modal formulas is defined inductively at points  $x$  in a topo-model  $M = (X, \mathcal{O}, \nu)$  in the following way:

$M, x \models p$	iff	$x \in \nu(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	there is $U \in \tau$ ( $x \in U$ and for all $y \in U$ : $M, y \models \varphi$ )
	and dually:	
$M, x \models \Diamond\varphi$	iff	for all $U \in \tau$ ( $x \in U \rightarrow$ there is $y \in U$ : $M, y \models \varphi$ )

# SOUND AND COMPLETE TOPO-AXIOMATIZATIONS

## Rules

(MP) if  $\vdash \varphi$  and  $\vdash \varphi \rightarrow \psi$ , then  $\vdash \psi$

(N) if  $\vdash \varphi$ , then  $\vdash \Box\varphi$

## Axioms

(K)  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(T)  $\Box\varphi \rightarrow \varphi$

(D)  $\Box\varphi \rightarrow \neg\Box\neg\varphi$

(4)  $\Box\varphi \rightarrow \Box\Box\varphi$

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S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

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S4=Topo

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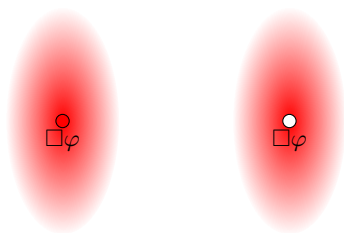


# WHAT ABOUT $T_d$ -SPACES (THE LEARNING SPACES)?

$T_d$  is not topo-definable.

Learnable spaces are not *topo*-definable.

Luckily, we can once again change the way we view  $\square$ .



# TOPOLOGICAL $d$ -SEMANTICS

## DEFINITION

Truth of modal formulas is defined inductively at points  $x$  in a topo-model  $M = (X, \tau, \nu)$  in the following way:

$M, x \models_d p$	iff	$x \in \nu(p)$ for each $p \in P$
$M, x \models_d \neg\varphi$	iff	not $M, x \models_d \varphi$
$M, x \models_d \varphi \wedge \psi$	iff	$M, x \models_d \varphi$ and $M, x \models_d \psi$
$M, x \models_d \Box\varphi$	iff	$\exists U \in \tau (x \in U \ \& \ \forall y \in U - \{x\} \ M, y \models_d \varphi)$

and dually:

$$M, x \models_d \Diamond\varphi \quad \text{iff} \quad \forall U \in \tau (x \in U \rightarrow \exists y \in U - \{x\} \ M, y \models_d \varphi)$$

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(w)  $(\varphi \wedge \Box\varphi) \rightarrow \Box\Box\varphi$

(GL)  $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$

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**KD45=DSO**

KD45 is the  $d$ -logic of DSO-spaces.

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GL=scattered

(GL)  $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$

GL is the  $d$ -logic of scattered spaces.

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## Axioms

(K)  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

wK4=Topo

(w)  $(\varphi \wedge \Box\varphi) \rightarrow \Box\Box\varphi$

wK4 is the  $d$ -logic of all topological spaces.

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## Axioms

(K)  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

$K4 = T_d$

(4)  $\Box\varphi \rightarrow \Box\Box\varphi$

Finally, K4 is the  $d$ -logic of all  $T_d$ -spaces.

## KD45 DOXASTIC $d$ -LOGIC (STEINVOLD 2006)

Because independent reasons (e.g., Stalnaker) one may want  $B := \Box$  to be:

$$(K) \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$(D) \quad \Box\varphi \rightarrow \neg\Box\neg\varphi$$

$$(4) \quad \Box\varphi \rightarrow \Box\Box\varphi$$

$$(5) \quad \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$$

THEOREM (STEINSVOLD 2006)

*KD45 is a sound and complete  $d$ -axiomatization of DSO spaces.*

DSO stands for 'derived sets are open'. DSO are  $T_d$ -spaces (by 4), in which all derived sets are open (5), except that there are no open singletons (D).



# QUESTIONS

But  $DSO \subset T_d$ .

So what do we talk about when we talk about beliefs in learning?

Should conjectures be interpreted as beliefs?

What if one restricts conjectures to only those which are 'proper' beliefs?

# OUTLINE

## INTRODUCTION

Learning and Solving in the Limit  
Topological Toolbox

## CHARACTERIZATION OF LEARNABILITY AND SOLVABILITY

## CONSTRUCTIVE ORDER-DRIVEN LEARNING

## TOWARDS EPISTEMIC LOGIC OF LEARNABILITY

Topological semantics  
Doxastic interpretations

## INTERMEDIATE (BUT INTERESTING ) CONCLUSIONS

# CONCLUSIONS

- ▶ Topological characterization of learnability & solvability in the limit.
- ▶ Universality of conditioning as a problem solving method.
- ▶ Use of stratification-like topological techniques.

Moreover:

- ▶ Learnable spaces are  $T_d$ .
- ▶  $T_d$ -spaces are not topo-definable.
- ▶ Learnability is not topo-definable.
- ▶ Learnability cannot be expressed by solely topo-definable belief operators.
- ▶ The existing topo- and  $d$ -logics of belief are too fluffy to capture learnability.

THANK YOU