Cofinal Elementary Cuts in Countable Models of Compositional Arithmetical Truth

Michał Tomasz Godziszewski
University of Warsaw

PhDs in Logic 2016
Darmstadt
May 10, 2016

Outline

Compositional Truth

Definition (Stratified Compositional Truth Theory)

Stratified Compositional Truth Theory (CT^{-}) *is an axiomatic theory obtained from PA by adjoining to it the following axioms:*

Compositional Truth

Definition (Stratified Compositional Truth Theory)

Stratified Compositional Truth Theory (CT^-) *is an axiomatic theory obtained from PA by adjoining to it the following axioms:*

1. $\forall s, t \in Trm[Tr(s = t) \equiv val(s) = val(t)]$,
2. $\forall x[Sent_{\mathcal{L}}(x) \Rightarrow (Tr(\neg x) \equiv \neg Tr(x))]$,
3. $\forall x, y[(Sent_{\mathcal{L}}(x \wedge y)) \Rightarrow (Tr(x \wedge y) \equiv Tr(x) \wedge Tr(y))]$,
4. $\forall x, y[(Sent_{\mathcal{L}}(x \vee y)) \Rightarrow (Tr(x \vee y) \equiv Tr(x) \vee Tr(y))]$,
5. $\forall v, x[Sent_{\mathcal{L}}(\forall vx) \Rightarrow (Tr(\forall vx) \equiv \forall t Tr(x(t/v)))]$,
6. $\forall v, x[Sent_{\mathcal{L}}(\exists vx) \Rightarrow (Tr(\exists vx) \equiv \exists t Tr(x(t/v)))]$,

where Tr *is a fresh unary predicate and where by* $Sent_{\mathcal{L}}(x)$ *we mean that* x *is the Gödel number af an arithmetical (without any occurence of the truth predicate) sentence of the arithmetical language* $\mathcal{L}$.

Types

Definition

- *A set* p *of the formulae of the language* $\mathcal{L}_M$ *(i.e. the language* $\mathcal{L}$ *extended with a constant name for every element of the model* $\mathcal{M}$*) with exactly one free variable* x *is* ***finitely satisfied*** *in* $\mathcal{M}$ *if and only if for any finite* $q \subset p$ *there exists an* $a \in M$ *such that for any* $\varphi(x) \in q$ $\mathcal{M} \models \varphi(a)$.

Types

Definition

- *A set p of the formulae of the language* $\mathcal{L}_M$ *(i.e. the language* $\mathcal{L}$ *extended with a constant name for every element of the model* $\mathcal{M}$*) with exactly one free variable x is* ***finitely satisfied*** *in* $\mathcal{M}$ *if and only if for any finite* $q \subset p$ *there exists an* $a \in M$ *such that for any* $\varphi(x) \in q$ $\mathcal{M} \models \varphi(a)$.
- *A* ***type*** *over a model* $\mathcal{M}$ *is a finitely satisfied set of formulae of the form* $\varphi(x, b)$ *with exactly one free variable x and at most one parameter* $b \in M$.

Types

Definition

- *A set p of the formulae of the language $\mathcal{L}_M$ (i.e. the language $\mathcal{L}$ extended with a constant name for every element of the model $\mathcal{M}$) with exactly one free variable x is **finitely satisfied** in $\mathcal{M}$ if and only if for any finite $q \subset p$ there exists an $a \in M$ such that for any $\varphi(x) \in q$ $\mathcal{M} \models \varphi(a)$.*
- *A **type** over a model $\mathcal{M}$ is a finitely satisfied set of formulae of the form $\varphi(x, b)$ with exactly one free variable x and at most one parameter $b \in M$.*
- *A type p over $\mathcal{M}$ is **recursive** if and only if the set of codes of formulae $\varphi(x, y)$ such that $\varphi(x, b) \in p$ is recursive.*

Types

Definition

- *A set p of the formulae of the language $\mathcal{L}_M$ (i.e. the language $\mathcal{L}$ extended with a constant name for every element of the model $\mathcal{M}$) with exactly one free variable x is **finitely satisfied** in $\mathcal{M}$ if and only if for any finite $q \subset p$ there exists an $a \in M$ such that for any $\varphi(x) \in q$ $\mathcal{M} \models \varphi(a)$.*
- *A **type** over a model $\mathcal{M}$ is a finitely satisfied set of formulae of the form $\varphi(x, b)$ with exactly one free variable x and at most one parameter $b \in M$.*
- *A type p over $\mathcal{M}$ is **recursive** if and only if the set of codes of formulae $\varphi(x, y)$ such that $\varphi(x, b) \in p$ is recursive.*
- *A type p over $\mathcal{M}$ is (globally) **realised** if and only if there exists an $a \in M$ such that for any $\varphi(x, b) \in p$ $\mathcal{M} \models \varphi(a, b)$.*

Recursive Saturation and Models of CT

Definition

Let $\mathcal{M}$ be a model of PA. A set $S \subseteq M$ is a ***full satisfaction class*** *for $\mathcal{M}$ if and only if $(\mathcal{M}, S) \models CT^-$.*

Recursive Saturation and Models of *CT*

Definition

Let $\mathcal{M}$ be a model of PA. A set $S \subseteq M$ is a ***full satisfaction class*** *for $\mathcal{M}$ if and only if $(\mathcal{M}, S) \models CT^{-}$.*

Definition

A model $\mathcal{M}$ of PA is ***recursively saturated*** *if and only if each recursive type over $\mathcal{M}$ is realised.*

Recursive Saturation and Models of *CT*

Definition

Let $\mathcal{M}$ be a model of PA. A set $S \subseteq M$ is a ***full satisfaction class*** *for $\mathcal{M}$ if and only if $(\mathcal{M}, S) \models CT^-$.*

Definition

A model $\mathcal{M}$ of PA is ***recursively saturated*** *if and only if each recursive type over $\mathcal{M}$ is realised.*

Theorem (A. Lachlan 1981)

If a nonstandard model $\mathcal{M} \models PA$ admits a full satisfaction class, then $\mathcal{M}$ is recursively saturated.

Recursive Saturation and Models of CT

Definition

*Let $\mathcal{M}$ be a model of PA. A set $S \subseteq M$ is a **full satisfaction class** for $\mathcal{M}$ if and only if $(\mathcal{M}, S) \models CT^-$.*

Definition

*A model $\mathcal{M}$ of PA is **recursively saturated** if and only if each recursive type over $\mathcal{M}$ is realised.*

Theorem (A. Lachlan 1981)

If a nonstandard model $\mathcal{M} \models PA$ admits a full satisfaction class, then $\mathcal{M}$ is recursively saturated.

Theorem (H. Kotlarski and S. Krajewski and A. Lachlan 1981, J. Barwise and J. Schlipf 1976, A. Enayat, and A. Visser 2013)

If a countable model $\mathcal{M} \models PA$ is recursively saturated, than it admits a full satisfaction class.

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

- *An* ***initial segment*** *of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{end} \mathcal{M}$, is such a subset $\mathcal{I} \subseteq M$ that is closed downwards, i.e. $\forall n \in \mathcal{I}$ and $\forall a \in M$ if $\mathcal{M} \models a < n$, then $a \in \mathcal{I}$.*

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

- *An **initial segment** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{end} \mathcal{M}$, is such a subset $\mathcal{I} \subseteq M$ that is closed downwards, i.e. $\forall n \in \mathcal{I}$ and $\forall a \in M$ if $\mathcal{M} \models a < n$, then $a \in \mathcal{I}$.*
- *An initial segment $\mathcal{I} \subseteq_{end} \mathcal{M}$ is a **cut** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{cut} \mathcal{M}$ if it is nonempty and closed under successor, i.e. $\forall n \in M\ (n \in \mathcal{I} \Rightarrow s(n) \in \mathcal{I})$.*

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

- *An* ***initial segment*** *of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{end} \mathcal{M}$, is such a subset $\mathcal{I} \subseteq M$ that is closed downwards, i.e. $\forall n \in \mathcal{I}$ and $\forall a \in M$ if $\mathcal{M} \models a < n$, then $a \in \mathcal{I}$.*
- *An initial segment $\mathcal{I} \subseteq_{end} \mathcal{M}$ is a* ***cut*** *of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{cut} \mathcal{M}$ if it is nonempty and closed under successor, i.e. $\forall n \in M\ (n \in \mathcal{I} \Rightarrow s(n) \in \mathcal{I})$.*
- *A cut $\mathcal{I} \subseteq_{cut} \mathcal{M}$ is a* ***proper cut*** *if $\mathcal{I} \neq M$.*

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

- *An **initial segment** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{end} \mathcal{M}$, is such a subset $\mathcal{I} \subseteq M$ that is closed downwards, i.e. $\forall n \in \mathcal{I}$ and $\forall a \in M$ if $\mathcal{M} \models a < n$, then $a \in \mathcal{I}$.*
- *An initial segment $\mathcal{I} \subseteq_{end} \mathcal{M}$ is a **cut** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{cut} \mathcal{M}$ if it is nonempty and closed under successor, i.e. $\forall n \in M\, (n \in \mathcal{I} \Rightarrow s(n) \in \mathcal{I})$.*
- *A cut $\mathcal{I} \subseteq_{cut} \mathcal{M}$ is a **proper cut** if $\mathcal{I} \neq M$.*
- *If $\mathcal{M}$ and $\mathcal{N}$ are models of PA, $\mathcal{M}$ is **cofinal** in $\mathcal{N}$, denoted $\mathcal{M} \subseteq_{cf} \mathcal{N}$, if for all $a \in N$ there exists $b \in M$ such that $\mathcal{N} \models a \leq b$.*

Cuts and Extensions

Definition

Let $\mathcal{M} \models PA$.

- *An **initial segment** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{end} \mathcal{M}$, is such a subset $\mathcal{I} \subseteq M$ that is closed downwards, i.e. $\forall n \in \mathcal{I}$ and $\forall a \in M$ if $\mathcal{M} \models a < n$, then $a \in \mathcal{I}$.*
- *An initial segment $\mathcal{I} \subseteq_{end} \mathcal{M}$ is a **cut** of $\mathcal{M}$, denoted $\mathcal{I} \subseteq_{cut} \mathcal{M}$ if it is nonempty and closed under successor, i.e. $\forall n \in M\, (n \in \mathcal{I} \Rightarrow s(n) \in \mathcal{I})$.*
- *A cut $\mathcal{I} \subseteq_{cut} \mathcal{M}$ is a **proper cut** if $\mathcal{I} \neq M$.*
- *If $\mathcal{M}$ and $\mathcal{N}$ are models of PA, $\mathcal{M}$ is **cofinal** in $\mathcal{N}$, denoted $\mathcal{M} \subseteq_{cf} \mathcal{N}$, if for all $a \in N$ there exists $b \in M$ such that $\mathcal{N} \models a \leq b$.*

Fact: any countable and nonstandard $\mathcal{M} \models PA$ has $2^{\aleph_0}$ proper cuts that are closed under $+, \times$.

Cofinal Extensions

Why study cofinal extensions in models of PA?

Cofinal Extensions

Why study cofinal extensions in models of *PA*?

> *A relatively neglected aspect of the study of nonstandard models of arithmetic is the study of their cofinal extensions. These extensions certainly do not present themselves to the intuition as readily as do their more popular cousins the end extensions; but they are not exactly shrouded in mystery or unnatural objects of study either. They are equal partners with end extensions in the construction of general extensions of models; they offer both special advantages and disadvantages worthy of our interest; and, occasionally, they are useful in understanding the generally more simply behaved end extensions. Cofinal extensions deserve more attention than they have traditionally received.*
> *(C. Smorynski, 1981)*

Gaps

The notion of a gap is a natural generalization of an idea of partitioning the universe of a nonstandard model into $\mathbb{Z}$-blocks around each element.

Gaps

The notion of a gap is a natural generalization of an idea of partitioning the universe of a nonstandard model into $\mathbb{Z}$-blocks around each element.

Definition

- *A* ***Skolem function*** *or term is a parameter-free definable PA-provably total function.*

Gaps

The notion of a gap is a natural generalization of an idea of partitioning the universe of a nonstandard model into $\mathbb{Z}$-blocks around each element.

Definition

- *A* ***Skolem function*** *or term is a parameter-free definable PA-provably total function.*
- *Let $\mathcal{F}$ be some family of Skolem terms $f : M \to M$ such that $\forall x, y \in M\ x < y \Rightarrow x \leq f(x) \leq f(y)$. There is a partition of M into sets, which we call $\mathcal{F}$-**gaps**. For any $a \in M$, $gap_{\mathcal{F}}(a)$ is the smallest set $C \subseteq M$ such that $a \in C$ and:*

$$\forall b \in C\ \forall f \in \mathcal{F}\ \forall x \in M\ \ (b \leq x \leq f(b)\ \vee\ x \leq b \leq f(x)) \Rightarrow x \in C.$$

Gaps

The notion of a gap is a natural generalization of an idea of partitioning the universe of a nonstandard model into $\mathbb{Z}$-blocks around each element.

Definition

- *A* ***Skolem function*** *or term is a parameter-free definable PA-provably total function.*
- *Let $\mathcal{F}$ be some family of Skolem terms $f : M \to M$ such that $\forall x, y \in M\; x < y \Rightarrow x \leq f(x) \leq f(y)$. There is a partition of M into sets, which we call $\mathcal{F}$-**gaps**. For any $a \in M$, $gap_{\mathcal{F}}(a)$ is the smallest set $C \subseteq M$ such that $a \in C$ and:*

$$\forall b \in C\, \forall f \in \mathcal{F}\, \forall x \in M\; (b \leq x \leq f(b) \vee x \leq b \leq f(x)) \Rightarrow x \in C.$$

- *The* ***gap of*** *$a \in M$, denoted by $gap(a)$, is the $\mathcal{F}$-gap of a, where $\mathcal{F}$ is the family of* ***all*** *such definable functions, i.e.*

$$\mathcal{F} = \{f : M \to M : f \text{ is Skolem and } \forall x, y \in M\; x < y \Rightarrow x \leq f(x) \leq f(y)\}.$$

Gaps

In other words:

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$*, let* $a \in M$*, and let* $A \subseteq M$*.*

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$, *let* $a \in M$, *and let* $A \subseteq M$.

- $inf(A) = \{x \in M : \forall y \in A\, x \leq y\}$,
- $sup(A) = \{x \in M : \exists y \in A\, x \leq y\}$,

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$*, let* $a \in M$*, and let* $A \subseteq M$*.*

- $inf(A) = \{x \in M : \forall y \in A\, x \leq y\}$,
- $sup(A) = \{x \in M : \exists y \in A\, x \leq y\}$,
- $Scl(a) = \{t(a) : \ t \text{ is a Skolem term of } PA\}$,

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$*, let* $a \in M$*, and let* $A \subseteq M$*.*

- $inf(A) = \{x \in M : \forall y \in A\, x \leq y\}$,
- $sup(A) = \{x \in M : \exists y \in A\, x \leq y\}$,
- $Scl(a) = \{t(a) : \ t \text{ is a Skolem term of } PA\}$,
- $\mathcal{M}(a) = \{x \in M : x < t(a) \text{ for some Skolem term } t\} = sup(Scl(a))$,

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$*, let* $a \in M$*, and let* $A \subseteq M$*.*

- $inf(A) = \{x \in M : \forall y \in A\, x \leq y\}$,
- $sup(A) = \{x \in M : \exists y \in A\, x \leq y\}$,
- $Scl(a) = \{t(a) :\ t \text{ is a Skolem term of } PA\}$,
- $\mathcal{M}(a) = \{x \in M : x < t(a) \text{ for some Skolem term } t\} = sup(Scl(a))$,
- $\mathcal{M}[a] = \{x \in M : t(x) < a \text{ for every Skolem term } t\}$,

Gaps

In other words:

Definition

let $\mathcal{M} \models PA$, let $a \in M$, and let $A \subseteq M$.

- $inf(A) = \{x \in M : \forall y \in A\, x \leq y\}$,
- $sup(A) = \{x \in M : \exists y \in A\, x \leq y\}$,
- $Scl(a) = \{t(a) : \ t \text{ is a Skolem term of } PA\}$,
- $\mathcal{M}(a) = \{x \in M : x < t(a) \text{ for some Skolem term } t\} = sup(Scl(a))$,
- $\mathcal{M}[a] = \{x \in M : t(x) < a \text{ for every Skolem term } t\}$,

The set $[a) = \mathcal{M}(a) \setminus \mathcal{M}[a]$ is exactly the gap(a).

Gaps and Cuts

It can be shown that:

- $\mathcal{M}(a)$ is the smallest elementary cut of $\mathcal{M}$ containing a, and
- $\mathcal{M}[a]$ is empty if and only if every elementary cut of $\mathcal{M}$ contains a.

Gaps and Cuts

It can be shown that:

- $\mathcal{M}(a)$ is the smallest elementary cut of $\mathcal{M}$ containing a, and
- $\mathcal{M}[a]$ is empty if and only if every elementary cut of $\mathcal{M}$ contains a.

Skolem functions are typically of the form

$$v \mapsto (min\ u)(\varphi(u, v)) \quad \text{or} \quad v \mapsto (max\ u)(\varphi(u, v)),$$

where $\varphi \in \mathcal{L}$.

Gaps and Cuts

It can be shown that:

- $\mathcal{M}(a)$ is the smallest elementary cut of $\mathcal{M}$ containing a, and
- $\mathcal{M}[a]$ is empty if and only if every elementary cut of $\mathcal{M}$ contains a.

Skolem functions are typically of the form

$$v \mapsto (min\ u)(\varphi(u, v)) \ \text{ or } \ v \mapsto (max\ u)(\varphi(u, v)),$$

where $\varphi \in \mathcal{L}$.
Notice if $x < y \in M[a]$, then for every Skolem function t,

$$t(x) \leq max\{t(y') : y' \leq y\} < d,$$

so $\mathcal{M}[a]$ is an initial segment of M.

Gaps and Cuts

It can be shown that:

- $\mathcal{M}(a)$ is the smallest elementary cut of $\mathcal{M}$ containing a, and
- $\mathcal{M}[a]$ is empty if and only if every elementary cut of $\mathcal{M}$ contains a.

Skolem functions are typically of the form

$$v \mapsto (min\ u)(\varphi(u,v)) \ \text{ or } \ v \mapsto (max\ u)(\varphi(u,v)),$$

where $\varphi \in \mathcal{L}$.
Notice if $x < y \in M[a]$, then for every Skolem function t,

$$t(x) \leq max\{t(y') : y' \leq y\} < d,$$

so $\mathcal{M}[a]$ is an initial segment of M.
Moreover, if $\mathcal{M}[a] \neq \emptyset$, then it is the biggest elementary cut of $\mathcal{M}$ which does not contain a.

Gaps and Cuts

It can be shown that:

- $\mathcal{M}(a)$ is the smallest elementary cut of $\mathcal{M}$ containing a, and
- $\mathcal{M}[a]$ is empty if and only if every elementary cut of $\mathcal{M}$ contains a.

Skolem functions are typically of the form

$$v \mapsto (min\ u)(\varphi(u, v)) \quad \text{or} \quad v \mapsto (max\ u)(\varphi(u, v)),$$

where $\varphi \in \mathcal{L}$.
Notice if $x < y \in M[a]$, then for every Skolem function t,

$$t(x) \leq max\{t(y') : y' \leq y\} < d,$$

so $\mathcal{M}[a]$ is an initial segment of M.
Moreover, if $\mathcal{M}[a] \neq \emptyset$, then it is the biggest elementary cut of $\mathcal{M}$ which does not contain a.
It follows that $gap(a)$ is always a convex subset of M containing a.

Short and Coshort Cuts

- If for some $a \in M$, $M = sup(gap(a))$, then we call $gap(a)$ the **last gap of** $\mathcal{M}$.

Short and Coshort Cuts

- If for some $a \in M$, $M = sup(gap(a))$, then we call $gap(a)$ the **last gap of** $\mathcal{M}$.
- A model with a last gap is called **short** – in other words, $\mathcal{M}$ is short if there is such an element $a \in M$ that its Skolem closure in $\mathcal{M}$ is cofinal in $\mathcal{M}$. Otherwise, $\mathcal{M}$ is **tall**.

Short and Coshort Cuts

- If for some $a \in M$, $M = sup(gap(a))$, then we call $gap(a)$ the **last gap of** $\mathcal{M}$.
- A model with a last gap is called **short** – in other words, $\mathcal{M}$ is short if there is such an element $a \in M$ that its Skolem closure in $\mathcal{M}$ is cofinal in $\mathcal{M}$. Otherwise, $\mathcal{M}$ is **tall**.
- If $\mathcal{M} \preceq_{cut} \mathcal{N}$ (i.e. $\mathcal{M}$ is an elementary cut of $\mathcal{N}$), we say that $\mathcal{M}$ is **short elementary cut** of $\mathcal{N}$ if $\mathcal{M}$ is short.

Short and Coshort Cuts

- If for some $a \in M$, $M = sup(gap(a))$, then we call $gap(a)$ the **last gap of** $\mathcal{M}$.
- A model with a last gap is called **short** – in other words, $\mathcal{M}$ is short if there is such an element $a \in M$ that its Skolem closure in $\mathcal{M}$ is cofinal in $\mathcal{M}$. Otherwise, $\mathcal{M}$ is **tall**.
- If $\mathcal{M} \preceq_{cut} \mathcal{N}$ (i.e. $\mathcal{M}$ is an elementary cut of $\mathcal{N}$), we say that $\mathcal{M}$ is **short elementary cut** of $\mathcal{N}$ if $\mathcal{M}$ is short.
- An elementary cut is **coshort** if $\mathcal{N} \setminus \mathcal{M}$ has the least gap, i.e. there is $a \in N \setminus M$ s.t. $M = inf(gap(a))$.

Diversity of (Elementary) Cuts

Let $\mathcal{N}$ be a countable recursively saturated (c.r.s) model of PA and let $\mathcal{E}(\mathcal{N})$ denote the family of its elementary submodels.

Diversity of (Elementary) Cuts

Let $\mathcal{N}$ be a countable recursively saturated (c.r.s) model of PA and let $\mathcal{E}(\mathcal{N})$ denote the family of its elementary submodels.

If every complete type realised in a countable model $\mathcal{M} \models PA$ is also realised in $\mathcal{N}$, then $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$.

Diversity of (Elementary) Cuts

Let $\mathcal{N}$ be a countable recursively saturated (c.r.s) model of *PA* and let $\mathcal{E}(\mathcal{N})$ denote the family of its elementary submodels.

If every complete type realised in a countable model $\mathcal{M} \models PA$ is also realised in $\mathcal{N}$, then $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$.

This implies that there are uncountably many nonisomorphic cuts in $\mathcal{E}(\mathcal{N})$.

Diversity of (Elementary) Cuts

Choice for elementary cuts is more limited:

Diversity of (Elementary) Cuts

Choice for elementary cuts is more limited:
if $\mathcal{M} \in \mathcal{E}(\mathcal{N})$ is a cut, then either

- $\mathcal{M}$ is tall, and in this case $\mathcal{M}$ is isomorphic to $\mathcal{N}$, or

Diversity of (Elementary) Cuts

Choice for elementary cuts is more limited:
if $\mathcal{M} \in \mathcal{E}(\mathcal{N})$ is a cut, then either

- $\mathcal{M}$ is tall, and in this case $\mathcal{M}$ is isomorphic to $\mathcal{N}$,
  or
- $\mathcal{M}$ is short, and then $\mathcal{M}$ is one of the countably many nonisomorphic elementary cuts of the form $sup(Scl(a))$.

Diversity of (Elementary) Cuts

Choice for elementary cuts is more limited:
if $\mathcal{M} \in \mathcal{E}(\mathcal{N})$ is a cut, then either

- $\mathcal{M}$ is tall, and in this case $\mathcal{M}$ is isomorphic to $\mathcal{N}$,
  or
- $\mathcal{M}$ is short, and then $\mathcal{M}$ is one of the countably many nonisomorphic elementary cuts of the form $sup(Scl(a))$.

How many short elementary cuts are there?

Diversity of (Elementary) Cuts

Choice for elementary cuts is more limited:
if $\mathcal{M} \in \mathcal{E}(\mathcal{N})$ is a cut, then either

- $\mathcal{M}$ is tall, and in this case $\mathcal{M}$ is isomorphic to $\mathcal{N}$,
  or
- $\mathcal{M}$ is short, and then $\mathcal{M}$ is one of the countably many nonisomorphic elementary cuts of the form $sup(Scl(a))$.

How many short elementary cuts are there?

Theorem (Smorynski, 1981)

Every countable recursively saturated model of PA has infinitely many pairwise nonisomorphic short elementary cuts

The isomorphism question for models with cuts

One of the interesting and natural questions conerning *pairs* for countable recursively saturated models of arithmetic and its elementary cuts is the following:

The isomorphism question for models with cuts

One of the interesting and natural questions conerning *pairs* for countable recursively saturated models of arithmetic and its elementary cuts is the following:

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $\mathcal{K}, \mathcal{K}'$ be elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?

The isomorphism question for models with cuts

One of the interesting and natural questions conerning *pairs* for countable recursively saturated models of arithmetic and its elementary cuts is the following:

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $\mathcal{K}, \mathcal{K}'$ be elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?

Another way to put it is: under what conditions, does the identity of theories of such pairs imply their isomoprhism?

State of the art

State of the art

Theorem (R. Kossak, H. Kotlarski, 1996)

Let $\mathcal{M}$ be a countable recursively saturated model of PA. Then for all $a, b \in \mathcal{M} \setminus \mathcal{M}(0)$, the strutures $(\mathcal{M}, \mathcal{M}[a])$ and $(\mathcal{M}, \mathcal{M}[b])$ are elementarily equivalent

State of the art

Theorem (R. Kossak, H. Kotlarski, 1996)

Let $\mathcal{M}$ be a countable recursively saturated model of PA. Then for all $a, b \in \mathcal{M} \setminus \mathcal{M}(0)$, the strutures $(\mathcal{M}, \mathcal{M}[a])$ and $(\mathcal{M}, \mathcal{M}[b])$ are elementarily equivalent

Definition (Set of complete types realised by a set)

For any $A \subseteq M$, $Tp(A) = \{tp^{\mathcal{M}}(a) : a \in A\}$

State of the art

Theorem (R. Kossak, H. Kotlarski, 1996)

Let $\mathcal{M}$ be a countable recursively saturated model of PA. Then for all $a, b \in \mathcal{M} \setminus \mathcal{M}(0)$, the strutures $(\mathcal{M}, \mathcal{M}[a])$ and $(\mathcal{M}, \mathcal{M}[b])$ are elementarily equivalent

Definition (Set of complete types realised by a set)

For any $A \subseteq M$, $Tp(A) = \{tp^{\mathcal{M}}(a) : a \in A\}$

Fact

Let $\mathcal{M} \models PA$ be c.r.s. Then, it is ω-homogenous.
Thus, for all $a, b \in M$, either $Tp(gap(a)) = Tp(gap(b))$ or $Tp(gap(a)) \cap Tp(gap(b)) = \emptyset$.

State of the art

Theorem (R. Kossak, J. Schmerl, 2012)

State of the art

Theorem (R. Kossak, J. Schmerl, 2012)

1. *Let $\mathcal{M} \models PA$ be a countable recursively saturated model. There are infinite sets L and U of gaps such that for distinct gaps γ and γ' in either L or U, $Tp(\gamma) \cap Tp(\gamma') = \emptyset$.*

State of the art

Theorem (R. Kossak, J. Schmerl, 2012)

1. *Let* $\mathcal{M} \models PA$ *be a countable recursively saturated model. There are infinite sets* L *and* U *of gaps such that for distinct gaps* γ *and* γ' *in either* L *or* U, $Tp(\gamma) \cap Tp(\gamma') = \emptyset$.
2. *If* γ *and* γ' *are the least gaps in* $\mathcal{M} \setminus \mathcal{K}$ *and* $\mathcal{M} \setminus \mathcal{K}'$, *respectively, such that* $Tp(\gamma) \cap Tp(\gamma') = \emptyset$, *then* $(\mathcal{M}, \mathcal{K}) \not\cong (\mathcal{M}, \mathcal{K}')$.

State of the art

Theorem (R. Kossak, J. Schmerl, 2012)

1. *Let $\mathcal{M} \models PA$ be a countable recursively saturated model. There are infinite sets L and U of gaps such that for distinct gaps γ and γ' in either L or U, $Tp(\gamma) \cap Tp(\gamma') = \emptyset$.*
2. *If γ and γ' are the least gaps in $\mathcal{M} \setminus \mathcal{K}$ and $\mathcal{M} \setminus \mathcal{K}'$, respectively, such that $Tp(\gamma) \cap Tp(\gamma') = \emptyset$, then $(\mathcal{M}, \mathcal{K}) \ncong (\mathcal{M}, \mathcal{K}')$.*
3. *There are infinitely many pairs of c.r.s. models $\mathcal{M} \models PA$ and coshort elementary cuts $\mathcal{K}$ and $\mathcal{K}'$ with (distinct) least gaps γ and γ' , respectively, such that $Tp(\gamma) \neq Tp(\gamma')$.*

State of the art

Theorem (R. Kossak, J. Schmerl, 2012)

1. *Let $\mathcal{M} \models PA$ be a countable recursively saturated model. There are infinite sets L and U of gaps such that for distinct gaps γ and γ' in either L or U, $Tp(\gamma) \cap Tp(\gamma') = \emptyset$.*
2. *If γ and γ' are the least gaps in $\mathcal{M} \setminus \mathcal{K}$ and $\mathcal{M} \setminus \mathcal{K}'$, respectively, such that $Tp(\gamma) \cap Tp(\gamma') = \emptyset$, then $(\mathcal{M}, \mathcal{K}) \not\cong (\mathcal{M}, \mathcal{K}')$.*
3. *There are infinitely many pairs of c.r.s. models $\mathcal{M} \models PA$ and coshort elementary cuts $\mathcal{K}$ and $\mathcal{K}'$ with (distinct) least gaps γ and γ' , respectively, such that $Tp(\gamma) \neq Tp(\gamma')$.*

Thus:
there are infinitely many elementarily equivalent and pairwise nonisomorphic pairs $(\mathcal{M}, \mathcal{K})$ with $\mathcal{K}$ being coshort.

Isomorphism question for short elementary cuts

Recall our question:

Isomorphism question for short elementary cuts

Recall our question:

Let $\mathcal{M} \models PA$ be a countable recursively saturated model, and let $\mathcal{K}, \mathcal{K}'$ be elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?

Isomorphism question for short elementary cuts

Recall our question:

*Let $\mathcal{M} \models PA$ be a countable recursively saturated model, and let $\mathcal{K}, \mathcal{K}'$ be **short** elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?*

Isomorphism question for short elementary cuts

Recall our question:

*Let $\mathcal{M} \models PA$ be a countable recursively saturated model, and let $\mathcal{K}, \mathcal{K}'$ be **short** elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?*

To provide a partial answer, we will use the following result of Smorynski:

Isomorphism question for short elementary cuts

Recall our question:

Let $\mathcal{M} \models PA$ be a countable recursively saturated model, and let $\mathcal{K}, \mathcal{K}'$ be ***short*** *elementary cuts of $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{K}) \equiv (\mathcal{M}, \mathcal{K}')$. Does it follow that $(\mathcal{M}, \mathcal{K}) \cong (\mathcal{M}, \mathcal{K}')$?*

To provide a partial answer, we will use the following result of Smorynski:

Theorem (Smorynski 1981)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $\mathcal{K} \in \mathcal{E}(\mathcal{M})$ be short (i.e. having a last gap). Then the following are definable without parameters in $(\mathcal{M}, \mathcal{K})$:

1. $\mathbb{N}$,
2. *the truth definition for $\mathcal{K}$,*
3. *the last (max) gap.*

A partial answer

We start with some easy definitions:

A partial answer

We start with some easy definitions:

Definition (Standard System)

A partial answer

We start with some easy definitions:

Definition (Standard System)

1. *Let $\mathcal{M} \models PA$ and let $a \in M$.* ***The set coded by** a **in** $\mathcal{M}$* *is the set*

$$S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\},$$

i.e. e.g. the set of non-zero indices of the binary expansion of the element a.

A partial answer

We start with some easy definitions:

Definition (Standard System)

1. *Let $\mathcal{M} \models PA$ and let $a \in M$.* ***The set coded by*** *a* ***in*** *$\mathcal{M}$ is the set*

$$S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\},$$

i.e. e.g. the set of non-zero indices of the binary expansion of the element a.

2. *$\mathcal{M}$* ***codes*** *$S \subseteq \mathbb{N}$ iff there is $a \in M$ such that a codes S in $\mathcal{M}$.*

A partial answer

We start with some easy definitions:

Definition (Standard System)

1. *Let $\mathcal{M} \models PA$ and let $a \in M$.* ***The set coded by** a **in** $\mathcal{M}$* *is the set*

$$S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\},$$

i.e. e.g. the set of non-zero indices of the binary expansion of the element a.

2. *$\mathcal{M}$ **codes** $S \subseteq \mathbb{N}$ iff there is $a \in M$ such that a codes S in $\mathcal{M}$.*
3. ***The standard system of** $\mathcal{M}$, denoted $SSy(\mathcal{M})$, is the set of all subsets of $\mathbb{N}$ that are coded in $\mathcal{M}$, i.e.*

$$SSy(\mathcal{M}) = \{S \subseteq \mathbb{N} : \exists a \in M \; S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\}\}.$$

A partial answer

We start with some easy definitions:

Definition (Standard System)

1. *Let $\mathcal{M} \models PA$ and let $a \in M$.* ***The set coded by** a **in** $\mathcal{M}$* *is the set*

$$S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\},$$

i.e. e.g. the set of non-zero indices of the binary expansion of the element a.

2. *$\mathcal{M}$* ***codes** $S \subseteq \mathbb{N}$* *iff there is $a \in M$ such that a codes S in $\mathcal{M}$.*
3. ***The standard system of** $\mathcal{M}$, denoted $SSy(\mathcal{M})$, is the set of all subsets of $\mathbb{N}$ that are coded in $\mathcal{M}$, i.e.*

$$SSy(\mathcal{M}) = \{S \subseteq \mathbb{N} : \exists a \in M\, S = \{n \in \mathbb{N} : \mathcal{M} \models (a)_n \neq 0\}\}.$$

Definition

Let $\mathcal{M} \models PA$. Then $Def(\mathcal{M})$ denotes the family of sets that are ***definable (with parameters) in** $\mathcal{M}$.*

A partial answer

When the standard system of a given model is relatively simple, than isomorphism of pairs follows, namely:

A partial answer

When the standard system of a given model is relatively simple, than isomorphism of pairs follows, namely:

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

A partial answer

When the standard system of a given model is relatively simple, than isomorphism of pairs follows, namely:

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

The proof uses the relativisation of $SSy(\mathcal{M})$ to definability in the standard model $\mathbb{N}$ - it enables us to describe the appropriate type.

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

Proof (Sketchy sketch of the idea)

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

Proof (Sketchy sketch of the idea)

By recursive saturation: $tp(a) \in SSy(\mathcal{M})$.

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

Proof (Sketchy sketch of the idea)

By recursive saturation: $tp(a) \in SSy(\mathcal{M})$.

Thus, $tp(a) \in Def(\mathbb{N})$, by assumption.

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

Proof (Sketchy sketch of the idea)

By recursive saturation: $tp(a) \in SSy(\mathcal{M})$.

Thus, $tp(a) \in Def(\mathbb{N})$, by assumption.

Suppose $\varphi(x)$ definse $tp(a)$ in $\mathbb{N}$. Then:

$$(\mathcal{M}, \mathcal{M}(a)) \models \exists v \in maxgap\, \forall x(\varphi(x) \Rightarrow Sat_{\mathcal{M}(a)}(x, v)),$$

which is possible by Smorynski's result.

A partial answer

Theorem (Tin Lok Wong, MTG)

Let $\mathcal{M} \models PA$ be a countable recursively saturated model and let $a, b \in M$ be nonstandard elements such that $(\mathcal{M}, \mathcal{M}(a)) \equiv (\mathcal{M}, \mathcal{M}(b))$.
If it holds that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$, then $(\mathcal{M}, \mathcal{M}(a)) \cong (\mathcal{M}, \mathcal{M}(b))$

Proof (Sketchy sketch of the idea)

By recursive saturation: $tp(a) \in SSy(\mathcal{M})$.

Thus, $tp(a) \in Def(\mathbb{N})$, by assumption.

Suppose $\varphi(x)$ definse $tp(a)$ in $\mathbb{N}$. Then:

$$(\mathcal{M}, \mathcal{M}(a)) \models \exists v \in maxgap\, \forall x(\varphi(x) \Rightarrow Sat_{\mathcal{M}(a)}(x, v)),$$

which is possible by Smorynski's result.

By the properties of c.r.s. models, this is enough for the isomorphism.

Summary and Questions

- Taking a nonstandard model of compositional truth such that all its coded sets are already definable in the standard model, we are able to identify isomorphic cofinal short elementary cuts of the model just by looking on the arithmetical theory of both pairs considered.

Summary and Questions

- Taking a nonstandard model of compositional truth such that all its coded sets are already definable in the standard model, we are able to identify isomorphic cofinal short elementary cuts of the model just by looking on the arithmetical theory of both pairs considered.
- The case that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$ is not a common one:
  If $PA \subseteq T$ - a complete consistent theory not in $Def(\mathbb{N})$, then $SSy(\mathcal{M}) \not\subseteq Def(\mathbb{N})$, by recursive saturation.

Summary and Questions

- Taking a nonstandard model of compositional truth such that all its coded sets are already definable in the standard model, we are able to identify isomorphic cofinal short elementary cuts of the model just by looking on the arithmetical theory of both pairs considered.
- The case that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$ is not a common one:
  If $PA \subseteq T$ - a complete consistent theory not in $Def(\mathbb{N})$, then $SSy(\mathcal{M}) \not\subseteq Def(\mathbb{N})$, by recursive saturation.
- However, if $\mathcal{M} \models Th(\mathbb{N})$, then $0^{\omega} \in Def_0(\mathcal{M}, \mathbb{N})$, (i.e. the ω-th jump of the Turing degree of recursive functions is definable without parameters in the pair $(\mathcal{M}, \mathbb{N})$, and $\mathbb{N}$ is *strong* in $\mathcal{M}$).

Summary and Questions

- Taking a nonstandard model of compositional truth such that all its coded sets are already definable in the standard model, we are able to identify isomorphic cofinal short elementary cuts of the model just by looking on the arithmetical theory of both pairs considered.
- The case that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$ is not a common one: If $PA \subseteq T$ - a complete consistent theory not in $Def(\mathbb{N})$, then $SSy(\mathcal{M}) \nsubseteq Def(\mathbb{N})$, by recursive saturation.
- However, if $\mathcal{M} \models Th(\mathbb{N})$, then $0^{\omega} \in Def_0(\mathcal{M}, \mathbb{N})$, (i.e. the ω-th jump of the Turing degree of recursive functions is definable without parameters in the pair $(\mathcal{M}, \mathbb{N})$, and $\mathbb{N}$ is *strong* in $\mathcal{M}$).
- It follows that f $\mathcal{M} \models Th(\mathbb{N})$ is countable and recursively saturated model such that $\mathcal{M}(a)$ is a short elementary cut of $\mathcal{M}$ and every $S \in SSy(\mathcal{M})$ is arithmetic in $0^{(\omega)}$, then there is an element $c \in gap(a)$ such that $tp(c) \in Def(\mathcal{M}, \mathcal{M}(a))$, which also gives a positive answer to our isomorphism question

Summary and Questions

- Taking a nonstandard model of compositional truth such that all its coded sets are already definable in the standard model, we are able to identify isomorphic cofinal short elementary cuts of the model just by looking on the arithmetical theory of both pairs considered.
- The case that $SSy(\mathcal{M} \subseteq Def(\mathbb{N})$ is not a common one: If $PA \subseteq T$ - a complete consistent theory not in $Def(\mathbb{N})$, then $SSy(\mathcal{M}) \nsubseteq Def(\mathbb{N})$, by recursive saturation.
- However, if $\mathcal{M} \models Th(\mathbb{N})$, then $0^{\omega} \in Def_0(\mathcal{M}, \mathbb{N})$, (i.e. the ω-th jump of the Turing degree of recursive functions is definable without parameters in the pair $(\mathcal{M}, \mathbb{N})$, and $\mathbb{N}$ is *strong* in $\mathcal{M}$).
- It follows that f $\mathcal{M} \models Th(\mathbb{N})$ is countable and recursively saturated model such that $\mathcal{M}(a)$ is a short elementary cut of $\mathcal{M}$ and every $S \in SSy(\mathcal{M})$ is arithmetic in $0^{(\omega)}$, then there is an element $c \in gap(a)$ such that $tp(c) \in Def(\mathcal{M}, \mathcal{M}(a))$, which also gives a positive answer to our isomorphism question
- Critical paths and further work: labelled gaps, minimal types, forcing, infinitary languages.

Thank you