Proof Mining: Proof Interpretations and Their Use in Mathematics

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Lecture I: General Introduction to the Unwinding of Proofs ('Proof Mining') and the Proof-Theoretic Methods

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Lecture II: Logical Metatheorems for Proof Mining and Applications

Lecture I

Early history

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Calibrate the contribution of the **use of ideal principles** in proofs of **real statements**.

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Shift of emphasis (G. Kreisel (1951): use proof-theoretic methods to **extract new information** from interesting proofs of existential statements.

'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?' (G. Kreisel)

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E.g. Let $\mathbf{C \equiv \forall x \in \mathbb{N} \exists y \in \mathbb{N}\, F(x, y)}$

Naive Attempt: try to extract an explicit computable function realizing (or bounding) '$\exists y$': $\mathbf{\forall x \in \mathbb{N}\, F(x, f(x))}$.

Naive attempt fails

Proposition

There exist a sentence $\mathbf{A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)}$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is **logical valid**,

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Proof: Take

$$\mathbf{A :\equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z)\big)},$$

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Any bound g on '$\exists y$', i.e. no computable g such that

$$\mathbf{\forall x \exists y \leq g(x) \forall z\, (T(x, x, y) \vee \neg T(x, x, z))}$$

since this would solve the halting problem! □

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Definition

$A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)$. Then the **Herbrand normal form** of A is defined as

$$A^H :\equiv \exists x_1, x_2 A_{qf}(x_1, f(x_1), x_2, g(x_1, x_2)),$$

where f, g are new function symbols, called index functions.

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A and A^H are equivalent with respect to logical validity, i.e.

$$\models A \Leftrightarrow \models A^H,$$

but are not logically equivalent (but only in the presence of AC).

We now consider again the sentence

$$\mathbf{A} \equiv \forall \mathbf{x}\, \exists \mathbf{y}\, \forall \mathbf{z}\, (\mathbf{P}(\mathbf{x}, \mathbf{y}) \vee \neg \mathbf{P}(\mathbf{x}, \mathbf{z})),$$

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$$\mathbf{A^H} \equiv \exists \mathbf{y} \big(\mathbf{P(x, y)} \vee \neg \mathbf{P(x, g(y))}\big)$$

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J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $\mathbf{A} \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)$. Then:

$\mathbf{PL} \vdash \mathbf{A}$ iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

$$\mathbf{A^{H,D}} :\equiv \bigvee_{i=1}^{k} \bigvee_{j=1}^{n} A_{qf}\left(s_i, f(s_i), t_j, g(s_i, t_j)\right)$$

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Note that the length of this disjunction is fixed: $k \cdot n$. The terms s_i, t_j can be extracted from a given PL-proof of A.

Herbrand's Theorem continued

Replacing in $A^{H,D}$ all terms '$g(s_i, t_j)$', '$f(s_i)$', by new variables (treating larger terms first) results in another tautological disjunction A^{Dis} s.t. A can be inferred from A by a **direct proof**.

An example

(Ulrich Berger) Consider the open first-order theory $\mathcal{T}$ in the language of first-order logic with equality and a constant 0 and two unary function symbols S, f. The only non-logical axiom of $\mathcal{T}$ is $\forall x(S(x) \neq 0)$.

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$\mathcal{T} \vdash \exists \mathbf{x}(\mathbf{f(S(f(x))) \neq x})$.

Proof: Suppose that

$$\forall \mathbf{x}(\mathbf{f(S(f(x))) = x}),$$

then f is injective, but also (since $S(x) \neq 0$) surjective on $\{x : x \neq 0\}$ and hence non-injective. Contradiction! □

Analyzing the above proof yields the following Herbrand terms:

$$\mathbf{PL} \vdash (\mathbf{S(s)} \neq \mathbf{0}) \rightarrow \bigvee_{\mathbf{j=1}}^{\mathbf{3}} (\mathbf{f(S(f(t_j))) \neq t_j}),$$

where

$$\mathbf{t_1 := 0, t_2 := f(0), t_3 := S(f(f(0))), s := f(f(0)).}$$

□

Remark

- For sentences $\mathbf{A} \equiv \forall x \exists y \forall z\, \mathbf{A_{qf}}(x, y, z)$, $\mathbf{A^{Dis}}$ can be written in the form

$$\mathbf{A_{qf}}(x, t_1, b_1) \vee \mathbf{A_{qf}}(x, t_2, b_2) \vee \ldots \vee \mathbf{A_{qf}}(x, t_k, b_k),$$

where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).

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  where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).
- Herbrand's theorem immediately extends to first-order theories $\mathcal{T}$ whose non-logical axioms $G_1, \ldots, G_n$ are all purely universal.

Theorem (Roth 1955)

An algebraic irrational number α has only finitely many exceptionally good rational approximations, i.e. for $\varepsilon > 0$ there are only finitely many $q \in \mathbb{N}$ such that

$$\mathbf{R(q) :\equiv q > 1 \wedge \exists! p \in \mathbb{Z} : (p, q) = 1 \wedge |\alpha - pq^{-1}| < q^{-2-\varepsilon}.}$$

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Theorem (Luckhardt 1985/89)

The following upper bound on $\#\{q : R(q)\}$ holds:

$$\mathbf{\#\{q : R(q)\} < \frac{7}{3}\varepsilon^{-1} \log N_\alpha + 6 \cdot 10^3 \varepsilon^{-5} \log^2 d \cdot \log(50\varepsilon^{-2} \log d),}$$

where $\mathbf{N_\alpha < \max(21 \log 2h(\alpha), 2 \log(1 + |\alpha|))}$ and h is the logarithmic absolute homogeneous height and $d = deg(\alpha)$.

Independently: Bombieri and van der Poorten 1988.

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- Techniques work only for restricted formal contexts: mainly purely universal ('algebraic') axioms, restricted use of induction, no higher analytical principles.

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- Require that one can 'guess' the correct Herbrand terms: in general procedure results in proofs of length $2_n^{|P|}$, where $2_{n+1}^k = 2^{2_n^k}$ (n cut complexity).

Towards generalizations of Herbrand's theorem

Allow **functionals $\Phi(x, f)$** instead of just Herbrand terms: Let's consider again the example

$$A \equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z))\big).$$

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$$\mathbf{A} \equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z))\big).$$

A^H can be realized by a computable functional of type level 2 which is defined by cases:

$$\Phi(x, g) := \begin{cases} c & \text{if } \neg T(x, x, g(c)) \\ g(c) & \text{otherwise.} \end{cases}$$

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From this definition it easily follows that

$$\forall x, g \big(T(x, x, \Phi(x, g)) \vee \neg T(x, x, g(\Phi(x, g)))\big).$$

Φ satisfies **G. Kreisel's no-counterexample interpretation!**

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

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Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} \forall \mathbf{i,j} \in \mathbf{[n; n+m]}\ (|\mathbf{a_i} - \mathbf{a_j}| \leq \mathbf{2^{-k}}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.

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where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.
Then the (partial) Herbrand normal form of this statement is

$$\mathbf{(2)}\ \forall \mathbf{k} \in \mathbb{N} \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{i,j} \in \mathbf{[n; n+g(n)]}\ (|\mathbf{a_i} - \mathbf{a_j}| \leq \mathbf{2^{-k}}).$$

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By contrast, there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (also referred to as **'metastability'** by T.Tao):

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Proposition

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall \mathbf{k} \in \mathbb{N} \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{n} \leq \Phi^*(\mathbf{g},\mathbf{k}) \forall \mathbf{i},\mathbf{j} \in [\mathbf{n};\mathbf{n}+\mathbf{g}(\mathbf{n})]\, (|\mathbf{a_i}-\mathbf{a_j}| \leq \mathbf{2^{-k}}),$$

where

$$\Phi^*(\mathbf{g},\mathbf{k}) := \tilde{\mathbf{g}}^{(\mathbf{2^k}-\mathbf{1})}(\mathbf{0}) \text{ with } \tilde{\mathbf{g}}(\mathbf{n}) := \mathbf{n} + \mathbf{g}(\mathbf{n}).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Remark

The previous result can be viewed as a polished form of a **Herbrand disjunction** of **variable (in k) length**:

$$\bigvee_{i=0}^{2^k-1} \left(|a_{\tilde{g}^{(i)}(0)} - a_{\tilde{g}(\tilde{g}^{(i)}(0))}| \leq 2^{-k}\right).$$

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Corollary (T. Tao's finite convergence principle)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 1 \geq a_0 \geq \ldots \geq a_M \geq 0 \exists N \in \mathbb{N}$$
$$\left(N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})\right).$$

One may take $M := \tilde{g}^{(2^k)}(0)$.

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad (\mathbf{n \geq 0})$$

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Theorem (von Neumann Mean Ergodic Theorem)

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

By logical metatheorems (see Lecture II tomorrow!):

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n \leq \Phi(\varepsilon, g, b, \eta)}\, \forall \mathbf{i, j \in [n; n + g(n)]}\, \big(\|\mathbf{A_i(x) - A_j(x)}\| < \varepsilon\big),$$

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X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)]\, \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^{(K)}(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \; h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

Computable rate of convergence iff the **norm of limit is computable**!

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

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As a corollary to our analysis of Birkhoff's proof, Avigad and Rute showed

Theorem (Avigad, Rute (ETDS 2015))

$(A_n(x))$ admits at most

$$\mathbf{2\log(M)} \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma} \cdot \mathbf{2\log(2M)} \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma}$$

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Partly possible because Birkhoff's proof only uses boundedly many (in the data) instances of the law-of-excluded-middle for $\exists$-statements!

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For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $\mathbf{C_n := \{0, 1, \ldots, n\}}$.

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Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N}\, \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n}\, \exists \mathbf{i} \leq \mathbf{n}\, \forall \mathbf{k} \in \mathbb{N}\, \exists \mathbf{m} \geq \mathbf{k}\, \big(\mathbf{f(m) = i}\big).$$

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Related problem: bad behavior w.r.t. modus ponens!

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Gödel's functional interpretation in five minutes

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- $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ are tuples of **functionals of finite type** over the base types of the system at hand.

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$\forall n, u, v \exists x, y\, (n{=}0 \to A_{Sh}(u,x)) \wedge (n{\neq}0 \to B_{Sh}(v,y))$
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$$\begin{cases} \mathbf{x}^* \gtrsim_{\mathbb{N}} \mathbf{x} :\equiv \mathbf{x}^* \geq \mathbf{x}, \\ \mathbf{x}^* \gtrsim_{\rho\to\tau} \mathbf{x} :\equiv \forall \mathbf{y}^*, \mathbf{y}(\mathbf{y}^* \gtrsim_{\rho} \mathbf{y} \to \mathbf{x}^*(\mathbf{y}^*) \gtrsim_{\tau} \mathbf{x}(\mathbf{y})). \end{cases}$$

Also relevant: **bounded functional interpretation** (F. Ferreira, P. Oliva)

Tao on a finitary approach to analysis

'it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on the one hand, and "soft", "qualitative", or "infinitary" analysis on the other hand.' ...'It is fairly well known that the results obtained by hard and soft analysis resp. can be connected to each other by various "correspondence principles" or "compactness principles". It is however my belief that the relationship between the two types of analysis is much deeper.' ...'There are rigorous results from proof theory which can allow one to automatically convert certain types of qualitative arguments into quantitative ones...'

(T. Tao: Soft analysis, hard analysis, and the finite convergence principle, 2007)

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4) Tao, T., Soft analysis, hard analysis, and the finite convergence principle. In: 'Structure and Randomness. AMS, 298pp., 2008'.

5) Special issue of 'Dialectica' on Gödel's interpretation with contributions e.g. by Ferreira, Kohlenbach, Oliva, 2008.

Lecture II

Recall from Lecture I: Gödel's functional interpretation

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

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where $\gtrsim$ is some suitable notion of being a 'bound' that applies to higher order function spaces (W.A. Howard)

$$\begin{cases} \mathbf{x}^* \gtrsim_{\mathbb{N}} \mathbf{x} :\equiv \mathbf{x}^* \geq \mathbf{x}, \\ \mathbf{x}^* \gtrsim_{\rho\to\tau} \mathbf{x} :\equiv \forall \mathbf{y}^*, \mathbf{y}(\mathbf{y}^* \gtrsim_{\rho} \mathbf{y} \to \mathbf{x}^*(\mathbf{y}^*) \gtrsim_{\tau} \mathbf{x}(\mathbf{y})). \end{cases}$$

Also relevant: **bounded functional interpretation** (F. Ferreira, P. Oliva)

General logical metatheorems I

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- Extraction of **bounds** from **noneffective** existence proofs.

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one can extract a closed term Φ of BA (+iteration)

$$\mathbf{BA\ (+\ IA\ )} \vdash \forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \exists \mathbf{m} \leq \mathbf{\Phi(f_x)}\, \mathbf{A_\exists(x, y, m).}$$

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$$\mathbf{BA\ (+\ IA\ )} \vdash \mathbf{\forall x \in P\, \forall y \in K\, \exists m \leq \Phi(f_x)\, A_\exists(x,y,m).}$$

Important:

$\Phi(f_x)$ does **not depend** on $y \in K$ but on a **representation** f_x of x!

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Compactness means constructively: **completeness** and **total boundedness**.

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Necessity of completeness: The set $[0,2]_{\mathbb{Q}}$ is totally bounded and constructively representable and

$$\mathbf{BA} \vdash \forall \mathbf{q} \in \mathbf{[0,2]_{\mathbb{Q}}}\ \exists \mathbf{n} \in \mathbb{N}(|\mathbf{q} - \sqrt{\mathbf{2}}| >_{\mathbb{R}} \mathbf{2^{-n}}).$$

However: **no uniform bound on** $\exists n \in \mathbb{N}$!

Necessity of total boundedness: Let B be the unit ball $C[0,1]$. B is bounded and constructively representable.

By Weierstraß' theorem

$$\mathbf{BA} \vdash \forall \mathbf{f} \in \mathbf{B} \exists \mathbf{n} \in \mathbb{N}\big(\mathbf{n} \text{ code of } \mathbf{p} \in \mathbb{Q}\mathbf{[X]} \text{ s.t. } \|\mathbf{p} - \mathbf{f}\|_\infty < \frac{\mathbf{1}}{\mathbf{2}}\big)$$

but **no uniform bound** on $\exists n$: take $f_n := \sin(nx)$.

Necessity of $A_\exists$ '$\exists$-formula':

Let (f_n) be the usual sequence of spike-functions in $C[0,1]$, s.t. (f_n) converges pointwise but not uniformly towards 0. Then

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Uniform bound only if $(f_n(x))$ **monotone** (Dini): '$\forall m \in \mathbb{N}$' **superfluous!**

Necessity of $\Phi(f_x)$ depending on a representative of x :

Consider

$$\mathbf{BA} \vdash \forall \mathbf{x} \in \mathbb{IR} \exists \mathbf{n} \in \mathbb{IN}(\mathbf{n} >_{\mathbb{IR}} \mathbf{x}).$$

Suppose there would exist an $=_{\mathbb{R}}$-extensional computable $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ producing such a n. Then Φ would represent a **continuous** and hence **constant** function $\mathbb{R} \to \mathbb{N}$ which gives a contradiction.

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

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MFI transforms **uniqueness statements**

$$\forall x \in P, y_1, y_2 \in K\Big(\bigwedge_{i=1}^{2} f(x, y_i) =_{\mathbb{R}} 0 \to d_K(y_1, y_2) =_{\mathbb{R}} 0\Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall x \in P, y_1, y_2 \in K, \varepsilon > 0\Big(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon\Big).$$

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Let $\widehat{y} \in K$ be the unique root of $f(x, \cdot)$, y_ε an ε-root $|f(x, y_\varepsilon)| < \varepsilon$.

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Then

$$d_K(\widehat{y}, y_{\Phi(x,\varepsilon)}) < \varepsilon).$$

Case study: strong unicity in L_1-approximation

P_n space of polynomials of degree $\leq n$, $f \in C[0,1]$,

$$\|f\|_1 := \int_0^1 |f(x)|dx, \quad \mathbf{dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1}.$$

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Best **approximation in the mean** of $f \in C[0,1]$ (Jackson 1926):

$$\forall f \in C[0,1] \exists! p_b \in P_n \left(\|f - p_b\|_1 = \mathbf{dist_1(f, P_n)} \right)$$

(existence **and** uniqueness use: WKL!)

Theorem (K./Paulo Oliva, APAL 2003)

Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\Psi(\omega, n, \varepsilon) := \min\{\tfrac{c_n \varepsilon}{8(n+1)^2}, \tfrac{c_n \varepsilon}{2} \omega_n(\tfrac{c_n \varepsilon}{2})\}, \text{ where}$$

$$c_n := \tfrac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2^{4n+3}(n+1)^{3n+1}} \text{ and}$$

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Then $\forall n \in \mathbb{N}, p_1, p_2 \in P_n$

$$\forall \varepsilon \in \mathbb{Q}^*_+ \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) \leq \Psi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon \Big).$$

Recall from I. Lecture

X **uniformly convex Banach space**, $f : X \to X$ **linear** and

$\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$. $\mathbf{A_n(x) := \frac{1}{n+1} \sum_{i=0}^{n} f^i(x)}$.

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$\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$. $\mathbf{A_n(x) := \frac{1}{n+1} \sum_{i=0}^{n} f^i(x)}$.

We extracted from Birkhoff's proof for the convergence of $(A_n(x))$ an effective bound Φ such that for all x with $\|x\| \leq b$:

$$\exists \mathbf{n \leq \Phi(\varepsilon, g, b, \eta)}\, \forall \mathbf{i, j \in [n; n + g(n)]}\, \big(\|\mathbf{A_i(x) - A_j(x)}\| < \varepsilon\big).$$

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Question: What is the reason for this strong uniformity and is there a logical **Metatheorem** to explain this?

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Question: What is the reason for this strong uniformity and is there a logical **Metatheorem** to explain this?

Answer: Yes! Crucial: **no separability assumption** on X is used.

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert spaces, abstract L^p- and $C(K)$-spaces, $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

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Condition: Defining axioms must have a monotone functional interpretation. This e.g. is the case if X is axiomatizable in positive bounded logic (Günzel/K., Adv. Math. 2016).

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Counterexamples (to extractibility of uniform bounds): for the classes of strictly convex ($\to$ uniformly convex) or separable ($\to$ totally bounded) spaces!

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathbf{PA}^{\omega,X}$ is the extension of Peano Arithmetic to all types over $\mathbb{N}, X$.

$\mathcal{A}^{\omega,X}$**:=**$\mathbf{PA}^{\omega,X}+\mathbf{DC}$, where

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{\mathsf{s} =_\rho \mathsf{t}}{\mathsf{r(s)} =_\tau \mathsf{r(t)}},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$\mathsf{s}^{\mathsf{X}} =_{\mathsf{X}} \mathsf{t}^{\mathsf{X}} :\equiv \mathsf{d_X(x,y)} =_{\mathbb{R}} \mathsf{0_{\mathbb{R}}},$$
$$\mathsf{s} =_{\rho\to\tau} \mathsf{t} :\equiv \forall \mathsf{v}^{\rho}(\mathsf{s(v)} =_\tau \mathsf{t(v)}).$$

A novel form of majorization

y, x functionals of types ρ and $\widehat{\rho} := \rho[\mathbb{N}/X]$:

$$\mathbf{x}^{\mathbb{N}} \gtrsim_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
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Example:

$$\mathbf{f}^* \gtrsim_{\mathbf{X}\to\mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \|\mathbf{x}\| \to \mathbf{f}^*(\mathbf{n}) \geq \|\mathbf{f}(\mathbf{x})].$$

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$f : X \to X$ is **nonexpansive (n.e.)** if $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|$.

Then $\lambda n.n + b \gtrsim_{X\to X} f$, if $b \geq \|f(0)\|$.

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Theorem

If $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ proves

$$\forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \forall \mathbf{z} \in \mathbf{X}\, \forall \mathbf{f} : \mathbf{X} \to \mathbf{X}\, \big(\mathbf{f} \text{ n.e.} \to \exists \mathbf{v} \in \mathbb{N}\, \mathbf{A}_\exists\big),$$

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then one can extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $x \in P, b \in \mathbb{N}$

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holds in **all nonempty (real) Hilbert space** X**.**

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Uniformly convex case: bound Φ additionally depends on a modulus of uniform convexity η.

Mean Ergodic Theorem again

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Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

X uniformly convex Banach space with modulus η and $f : X \to X$ nonexpansive linear operator. Let $b > 0$. Then there is an effective functional Φ in ε, g, b, η s.t. for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}, \mathbf{n} + \mathbf{g(n)}]\, \left(\|\mathbf{A_i(x)} - \mathbf{A_j(x)}\| < \varepsilon\right).$$

(see Lecture I)

A theorem of R. Wittmann

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Halpern iterations: $U : C \to C$ nonexpansive, $u_0 \in C$, $\alpha_n \in [0,1]$

$$\mathbf{u_{n+1} := \alpha_{n+1}\, u_0 + (1 - \alpha_{n+1})\, U(u_n).}$$

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Theorem (R. Wittmann, 1992): $C \subseteq X$ closed and convex, $u_0 \in C$ and $Fix(U) \neq \emptyset$. Under suitable conditions on (α_n) (satisfied e.g. for $\alpha_n := \frac{1}{n+1}$) (u_n) converges strongly towards the fixed point of U that is closest to u_0.

A theorem of R. Wittmann

Halpern iterations: $U : C \to C$ nonexpansive, $u_0 \in C$, $\alpha_n \in [0,1]$

$$\mathbf{u_{n+1} := \alpha_{n+1}\, u_0 + (1 - \alpha_{n+1})\, U(u_n).}$$

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Remark: Wittmann's theorem is a **nonlinear generalization of the Mean ergodic theorem**: for $\alpha_n := 1/(n+1)$, $C := X$ and **linear** U, the Halpern iteration coincides with the Cesàro means. Hence the Mean Ergodic Theorem follows as a special case.

General features of the logical analysis of Wittmann's

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- Use of weak compactness gets in the end **eliminated** via a quantitative projection argument and a profound use of the power of majorizability.
- As a consequence, both proofs yield ordinary **primitive recursive bounds** with **elementary verifications**.
- **Quadratic rate** of **asymptotic regularity** for $1/(n+1)$ (K. Adv. Math. 2011):

$$\forall \mathbf{n} \in \mathbb{N}\, \forall \mathbf{k} \geq \mathbf{4dn(8dn+2)}\ \left(\|\mathbf{u_k} - \mathbf{U(u_k)}\| \leq \frac{\mathbf{1}}{\mathbf{n}}\right),$$

where $\mathbf{d} \geq \mathbf{diam(C)}$.

A quantitative metastable version of Wittmann's theorem

Theorem (K., Adv. Math. 2011)

Let $\alpha_n := 1/(n+1)$ and (u_n) as above. Then for $\varepsilon \in (0,1)$

$$\forall \mathbf{g} : \mathbb{N}^{\mathbb{N}} \exists \mathbf{k} \leq \Phi(\varepsilon/2, \mathbf{g}^{+}, \mathbf{d})\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{k}; \mathbf{k} + \mathbf{g}(\mathbf{k})]\ \left(\|\mathbf{u_i} - \mathbf{u_j}\| \leq \varepsilon\right),$$

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where

$\Phi(\varepsilon, g, d) := \rho(\varepsilon^2/4d^2, \chi_{d,\varepsilon}(N_{\varepsilon,g,d}))$ with

$N_{\varepsilon,g,d} := 16d \cdot \left(\max\left\{(\Delta^*_{\varepsilon,g})^{(i)}(1) : i \leq n_{\varepsilon,d}\right\}\right)^2$, $n_{\varepsilon,d} := \left\lceil \frac{d^2}{\varepsilon_d} \right\rceil$,

$\varepsilon_d := \frac{\varepsilon^4}{8192d^2,}$ $\Delta^*_{\varepsilon,g}(n) := \lceil 1/\Omega_d(\varepsilon/2, \tilde{g}^M, \chi_{d,\varepsilon}(16d \cdot n^2)) \rceil$,

with $\Omega_d(\varepsilon, g, j) := \delta_{\varepsilon, \tilde{g}(\rho(\varepsilon^2/2d^2, j))}$, where $\delta_{\varepsilon,m} := \frac{\varepsilon^2}{16dm}$,

$\rho(\varepsilon, n) := \left\lceil \frac{n+1}{\varepsilon} \right\rceil$, $\chi_{d,\varepsilon}(n) := \max\left\{\chi_d(n), \left\lceil \frac{32d^2}{\varepsilon^2} \right\rceil\right\}$,

$\chi_d(n) := 4dn(4dn+2)$, $\tilde{g}(n) := \max\{n, g(n)\}$ and $g^+(n) := n + g(n)$.

'Proof Mining' in core mathematics

- During the last 20 years this proof-theoretic approach has resulted in **numerous new quantitative results** as well as **qualitative uniformity results** in nonlinear analysis: fixed point theory ($\geq$40), ergodic theory ($\geq$15), optimization (D. Körnlein) ($\geq$5), topological dynamics ($\geq$ 5), approximation theory ($\geq$ 5), abstract Cauchy problems (A. Koutsoukou-Argyraki) (2) etc.

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- Proof mining has also led to **new concepts** that are now commonly used in analysis: *W*-**hyperbolic spaces** (K.2005), *UCW*-**hyperbolic spaces** (Leuștean 2007), (generalized) **uniform Fejér monotonicity** (Leuștean/Nicolae/K. 2014).

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'We shall establish Theorem 1.6 by "finitary ergodic theory" techniques, reminiscent of those used in [Green-Tao]...' 'The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit'...'In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation'
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

2016 survey:

www.mathematik.tu-darmstadt.de/~kohlenbach/progress.pdf

2008 book: