Towards Π_2-cut-introduction

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FAKULTÄT FÜR !NFORMATIK

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logics Logical Methods in Computer Science

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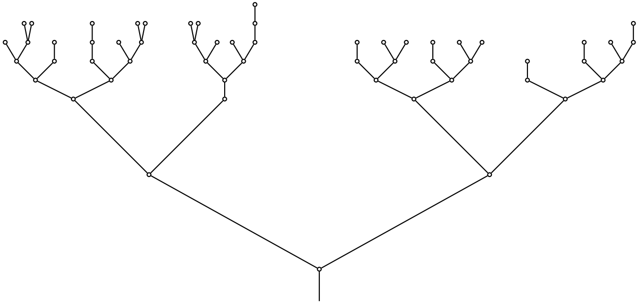
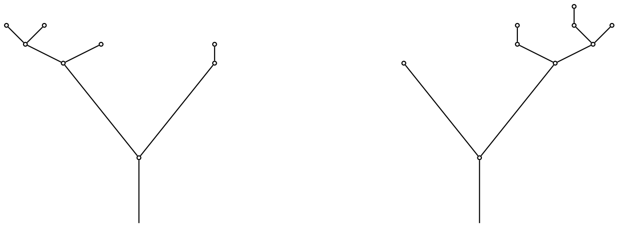
Complex proof tree



$\forall \bar{x} F(\bar{x}) \vdash$

Simple proof tree



$$\frac{\forall \bar{x} F(\bar{x}) \vdash C \qquad C, \forall \bar{x} F(\bar{x}) \vdash}{\forall \bar{x} F(\bar{x}) \vdash}$$

A brief description of the sequent calculus

$$\forall x.P(r, fx), \forall x.Q(r, gx) \vdash \exists x, y, z.P(x, y) \land Q(x, z)$$

$$(\forall x.P(r, fx) \land \forall x.Q(r, gx)) \to \exists x, y, z.Q(x, fy) \land P(fy, z)$$

A brief description of the sequent calculus

$$\frac{}{\forall x.P(r, fx), \forall x.Q(r, gx) \vdash \exists x, y, z.P(x, y) \land Q(x, z)}$$

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$$\dfrac{\dfrac{P(r, f\,c), Q(r, g\,c) \vdash P(r, fc) \wedge Q(r, gc)}{\vdots}}{\forall x.P(r, fx), \forall x.Q(r, gx) \vdash \exists x, y, z.P(x, y) \wedge Q(x, z)}$$

Midsequent theorem

Herbrand termset

A brief description of the sequent calculus

$$\dfrac{\dfrac{P(r, fc), Q(r, gc) \vdash P(r, fc) \qquad P(r, fc), Q(r, gc) \vdash Q(r, gc)}{P(r, fc), Q(r, gc) \vdash P(r, fc) \wedge Q(r, gc)}}{\dfrac{\vdots}{\forall x.P(r, fx), \forall x.Q(r, gx) \vdash \exists x, y, z.P(x, y) \wedge Q(x, z)}}$$

Subformula property
Active formula

The cut rule in sequent calculus

$$\text{cut}\ \frac{\Delta \vdash \Gamma, A \qquad A, \Delta \vdash \Gamma}{\Delta \vdash \Gamma}$$

- The cut rule represents the use of a lemma (Therefore we can call cut-introduction "lemma generation").

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- It does not fulfill the subformula property.

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$$\text{cut}\ \frac{\Delta \vdash \Gamma, A \qquad A, \Delta \vdash \Gamma}{\Delta \vdash \Gamma}$$

- The cut rule represents the use of a lemma (Therefore we can call cut-introduction "lemma generation").
- It does not fulfill the subformula property.
- It is admissible.

The extended Herbrand sequent for a Π_1-cut-formula

Π_1-case:

$\forall x.F(x)$ where F is quantifier free.

The extended Herbrand sequent for a Π_1-cut-formula

Π_1-case:

$$\frac{\Delta \ \vdash \Gamma, \forall x.F(x) \qquad\qquad \forall x.F(x), \Delta \ \vdash \Gamma}{\Delta \vdash \Gamma}$$

The extended Herbrand sequent for a Π_1-cut-formula

Π_1-case:

$$\dfrac{\dfrac{\Delta' \vdash \Gamma', F(\alpha)}{\Delta \vdash \Gamma, \forall x.F(x)} \qquad \dfrac{F(t_1), \ldots, F(t_p), \Delta'' \vdash \Gamma''}{\forall x.F(x), \Delta \vdash \Gamma}}{\Delta \vdash \Gamma}$$

where $\Delta', \Gamma', \Delta''$, and Δ'' are quantifier-free.

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The extended Herbrand sequent for a Π_1-cut-formula

Π_1-case:

$$\dfrac{\dfrac{\Delta' \vdash \Gamma', F(\alpha)}{\Delta' \vdash \Gamma', F(\alpha)} \qquad \dfrac{F(t_1), \ldots, F(t_p), \Delta'' \vdash \Gamma''}{F(t_1) \wedge \ldots \wedge F(t_p), \Delta'' \vdash \Gamma''}}{\color{red}{F(\alpha) \rightarrow F(t_1) \wedge \ldots \wedge F(t_p)}, \Delta', \Delta'' \vdash \Gamma', \Gamma''}$$

The extended Herbrand sequent for a Π_1-cut-formula

Π_1-case:

$$F(\alpha) \to F(t_1) \wedge \ldots \wedge F(t_p), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

where α does not occur in $t_1, \ldots, t_p$.

The extended Herbrand sequent for a Π_2-cut-formula

Π_2-case:

$\forall x \exists y. F(x, y)$ where F is quantifier free.

The extended Herbrand sequent for a Π_2-cut-formula

Π_2-case:

$$\dfrac{\dfrac{\Delta \vdash \Gamma, \exists y.F(\alpha, y)}{\Delta \vdash \Gamma, \forall x \exists y.F(x,y)} \qquad \dfrac{\dfrac{F(t_1, \beta_1), \forall x \exists y F(x,y), \Delta \vdash \Gamma}{\exists y F(t_1, y), \forall x \exists y F(x,y), \Delta \vdash \Gamma}}{\forall x \exists y.F(x,y), \Delta \vdash \Gamma}}{\Delta \vdash \Gamma}$$

with $\mathrm{FV}(r_i) \subseteq \{\alpha\}$ for all i and
$\mathrm{FV}(t_j) \subseteq \{\beta_1, \ldots, \beta_{j-1}\}$ for $j \geq 2$.

The extended Herbrand sequent for a Π_2-cut-formula

Π_2-case:

$$\dfrac{\dfrac{\dfrac{\Delta' \vdash \Gamma', F(\alpha, r_1), \ldots, F(\alpha, r_m)}{\Delta \vdash \Gamma, \exists y.F(\alpha, y)}}{\Delta \vdash \Gamma, \forall x \exists y.F(x, y)} \qquad \dfrac{\dfrac{\dfrac{F(t_1, \beta_1), \ldots, F(t_p, \beta_p), \Delta'' \vdash \Gamma''}{F(t_1, \beta_1), \forall x \exists y F(x, y), \Delta \vdash \Gamma}}{\exists y F(t_1, y), \forall x \exists y F(x, y), \Delta \vdash \Gamma}}{\forall x \exists y.F(x, y), \Delta \vdash \Gamma}}{\Delta \vdash \Gamma}$$

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The extended Herbrand sequent for a Π_2-cut-formula

Π_2-case:

$$\bigvee_{i=1}^{m} F(\alpha, r_i) \rightarrow \bigwedge_{j=1}^{p} F(t_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

where $\mathrm{FV}(r_i) \subseteq \{\alpha\}$ for all i,
$\mathrm{FV}(t_i) \subseteq \{\beta_1, \ldots, \beta_{j-1}\}$ for $j \geq 2$, and $\mathrm{FV}(t_1) = \emptyset$.

Schematic Π_2-grammar

A schematic Π_2-grammar G is a tuple $\langle \tau, N, \Sigma, \mathrm{Pr} \rangle$ where

- τ is the starting symbol.
- N are the (rigid) non-terminals $\{\tau, \alpha, \beta_1, \ldots, \beta_p\}$.

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- Pr are the production rules

$$\begin{gathered}\{\alpha \to t_1 \mid, \ldots, \mid t_p\} \cup \\ \{\beta_p \to r_1[\alpha \backslash t_p] \mid, \ldots, \mid r_m[\alpha \backslash t_p]\} \cup \\ \vdots \\ \{\beta_1 \to r_1[\alpha \backslash t_1] \mid, \ldots, \mid r_m[\alpha \backslash t_1]\} \cup \\ \{\tau \to u_{1,\alpha}, \ldots, \tau \to u_{k,\alpha}, \tau \to u_{1,\beta}, \ldots, \tau \to u_{n,\beta}\}.\end{gathered}$$

Extended Herbrand-sequent

$$\bigvee_{i=1}^{m} F(\alpha, r_i) \to \bigwedge_{j=1}^{p} F(t_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

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- Pr are the production rules

$$\begin{gathered}
\{\alpha \rightarrow t_1 \mid , \ldots , \mid t_p\} \cup \\
\{\beta_p \rightarrow r_1[\alpha \backslash t_p] \mid , \ldots , \mid r_m[\alpha \backslash t_p]\} \cup \\
\vdots \\
\{\beta_1 \rightarrow r_1[\alpha \backslash t_1] \mid , \ldots , \mid r_m[\alpha \backslash t_1]\} \cup \\
\{\tau \rightarrow u_{1,\alpha}, \ldots, \tau \rightarrow u_{k,\alpha}, \tau \rightarrow u_{1,\beta}, \ldots, \tau \rightarrow u_{n,\beta}\}.
\end{gathered}$$

The grammar must be acyclic.
$\tau > \alpha > \beta_p > \ldots > \beta_1$
The derivations must fulfil the rigidity condition.

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- we assume a provable sequent
- where the Herbrand termset can be represented by a grammar G
- and search a cut-formula to prove the sequent via a cut rule.

We assume that the language of the grammar is a superset of the termset.

The schematic extended Herbrand sequent

Let G be a schematic Π_2-grammar as before. The schematic extended Herbrand sequent is defined as

$$\bigvee_{i=1}^{m} X(\alpha, r_i) \to \bigwedge_{j=1}^{p} X(t_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

where X is a second-order variable.

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where X is a second-order variable.
We call $\lambda x, y.S(x, y)$ a solution if

$$\bigvee_{i=1}^{m} S(\alpha, r_i) \rightarrow \bigwedge_{j=1}^{p} S(t_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

is a tautology.

Results thus far obtained

- In contrast to Π_1-cut-introduction the inversion procedure is not complete for Π_2-cut-introduction.
  In the Π_2-case we will not find for each superset of the termset a cut-formula, while we will always find a cut-formula in the Π_1-case.

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  In the Π_2-case we will not find for each superset of the termset a cut-formula, while we will always find a cut-formula in the Π_1-case.
- We have defined a large and relevant class of Π_2-cut-formulae A for which we can decide whether there is a proof with cut (the cut-formula only interact with the context).

Results thus far obtained

Let

$$H(X) := \bigvee_{i=1}^{p} X(\alpha, t_i) \rightarrow \bigwedge_{j=1}^{m} X(r_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

be a schematic extended Herbrand sequent and S be a formula such that

$$H(S) := \bigvee_{i=1}^{p} S(\alpha, t_i) \rightarrow \bigwedge_{j=1}^{m} S(r_j, \beta_j), \Delta', \Delta'' \vdash \Gamma', \Gamma''$$

is a tautology. Assume a maximal **G3c**-derivation of $H(S)$. We say S is a balanced solution if in all axioms of $H(S)$ at least one of the active formulas is not an ancestor of S.

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- We have defined a large and relevant class of Π_2-cut-formulae A for which we can decide whether there is a proof with cut (the cut-formula only interact with the context).
- We also provide a method to compute these cut-formulae (G^*-unifiability).

Results thus far obtained

$\mathcal{G}^*$-unifiability is a unification method to unify literals corresponding to a schematic Π_2-grammar $\mathcal{G}$. Assume a maximal **G3c**-derivation of the context $\Delta', \Delta'' \vdash \Gamma', \Gamma''$ of a schematic extended Herbrand sequent and the non-tautological leaf

$$P(\alpha, t_1\alpha) \vdash P(r_2, \beta_2).$$

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$$P(\alpha, t_1\alpha) \vdash P(r_2, \beta_2).$$

Given that the schematic Π_2-grammar contains the production rules $\alpha \to r_2$ and $\beta_2 \to t_1 r_2$ we can $\mathcal{G}^*$-unify $P(\alpha, t_1\alpha)$ and $P(r_2, \beta_2)$ to

$$P(x, y)$$

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- We have defined a large and relevant class of Π_2-cut-formulae A for which we can decide whether there is a proof with cut (the cut-formula only interact with the context).
- We also provide a method to compute these cut-formulae (G^*-unifiability).
- We get an exponential compression.

Further research questions

- Implement the method.
- Prove the decidability of the solvability of Π_2-cut-introduction.
- We want to introduce many Π_2-cuts and Π_2-cuts with blocks of quantifiers.
- Investigate Π_n-cut-introduction with $n \geq 3$.