# Definability of Cai-Fürer-Immerman Problems in Choiceless Polynomial Time

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RWTH Aachen

PHDs in Logic VIII, Darmstadt 2016

Logics capturing complexity classes

# Motivation: Database Theory

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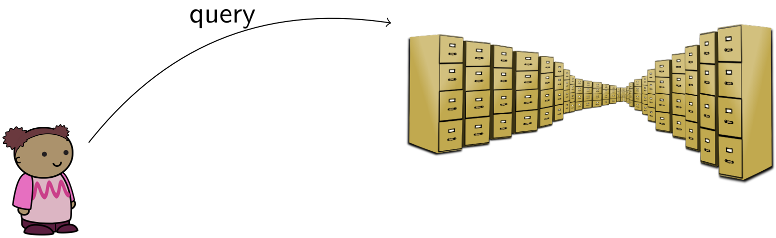

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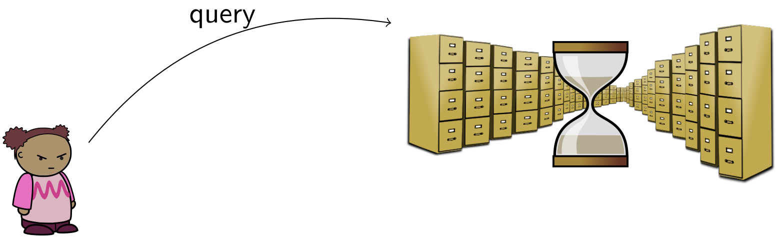

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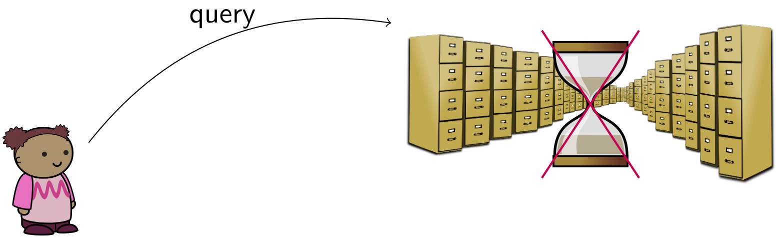

Logics capturing complexity classes

Question (Chandra, Harel 1982)

Is there a database query language expressing exactly the efficiently computable queries?

Logics capturing complexity classes

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Logics capturing complexity classes

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**Question (Gurevich 1988)**

Is there a logic capturing $\textsc{Ptime}$?

**Theorem (Fagin)**

$\exists\mathrm{SO}$ *captures NP.*

connected to “P = NP?”

# Candidates for a logic capturing PTIME

PTIME

$\subsetneq$

FO    cannot define transitive closure

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$\textsc{Ptime}$

$\cup\!\!\!\!\!\neq$

FP captures $\textsc{Ptime}$ on ordered structures

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| | |
|---|---|
| PTIME | |
| $\subsetneq$ | |
| FPC | captures PTIME on many interesting classes of structures. |
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| FP | captures PTIME on ordered structures |
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| FO | cannot define transitive closure |

# Candidates for a logic capturing PTIME

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$\mathrm{FP} + \mathsf{rk} \qquad \mathrm{CPT} + \mathrm{C}$

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**Theorem (Dawar,Richerby,Rossman)**

*The Cai-Fürer-Immerman query over ordered graphs is* $\mathrm{CPT}$*-definable.*

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Iterated creation of hereditarily finite sets,
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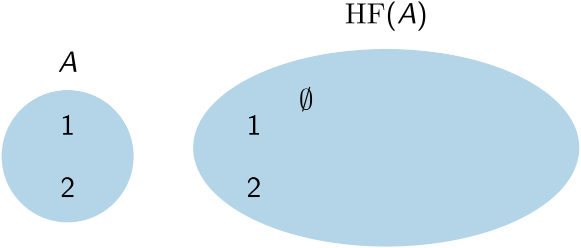
## Hereditarily finite sets



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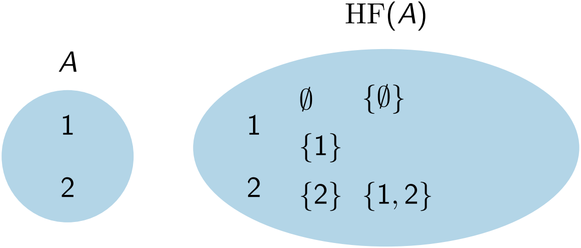

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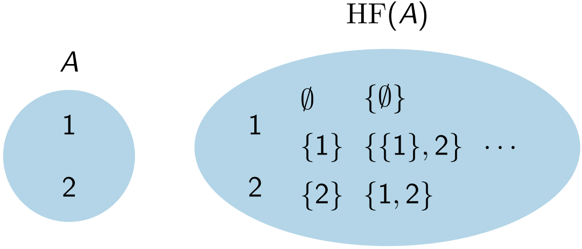

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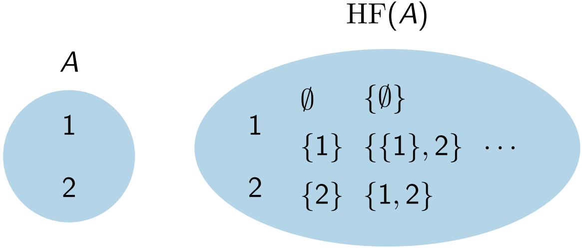


## Operations

- Set-theoretic operations ($\emptyset$, $\in$, $\cup$,...), boolean connectives
- Comprehension terms: $\{s(x) : x \in t : \varphi(x)\}$

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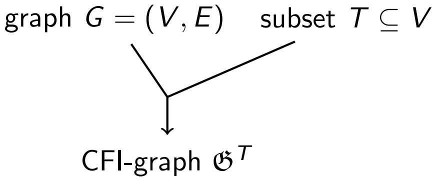


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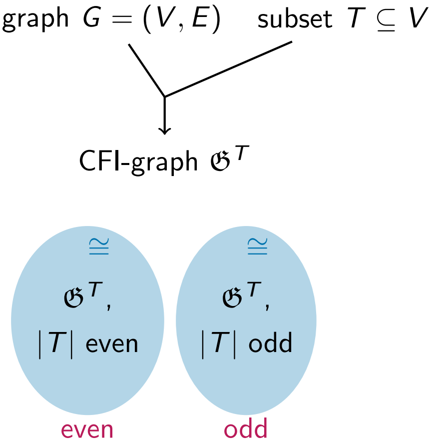
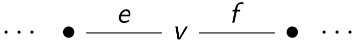


# Cai-Fürer-Immerman graphs: construction

Graph $\mathfrak{G}$

vertex $v$
$\rightsquigarrow$ gadget $v^*$



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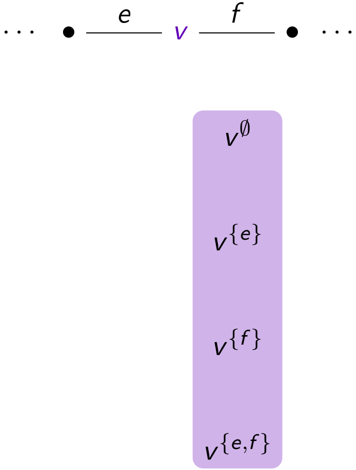

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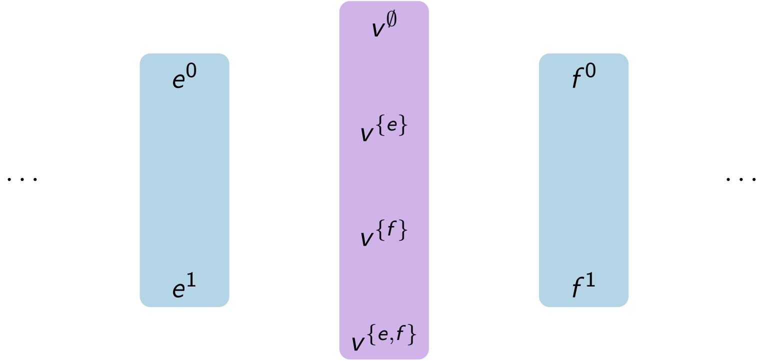

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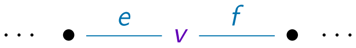


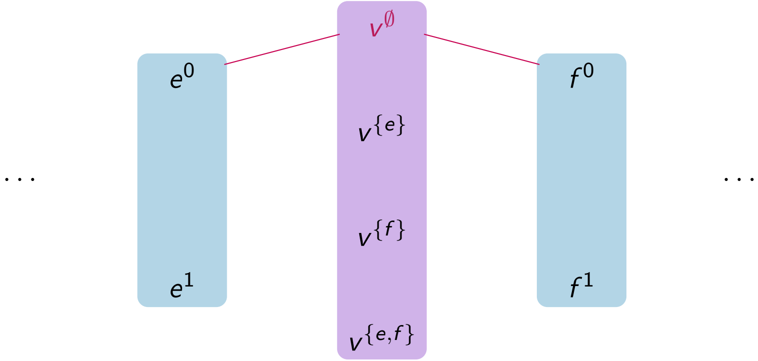

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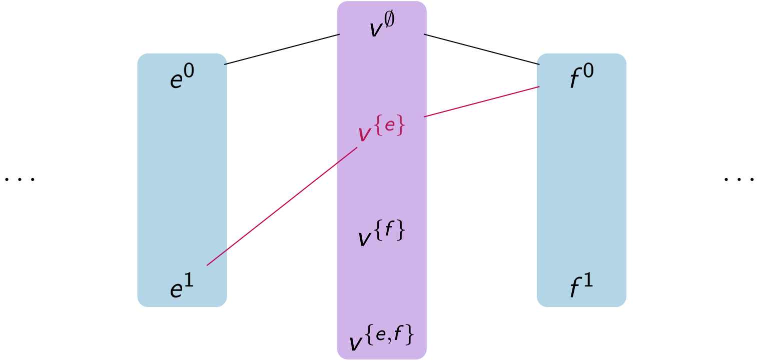

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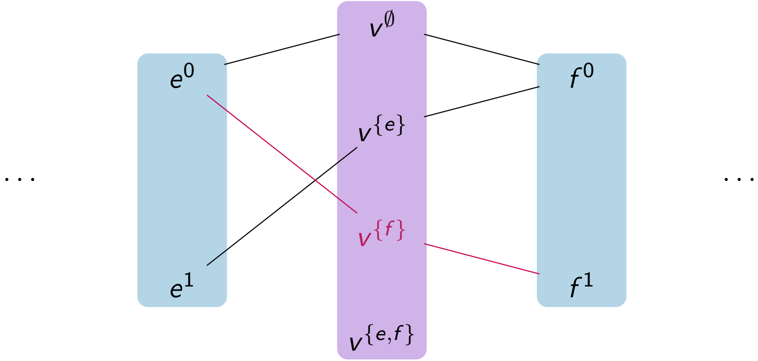

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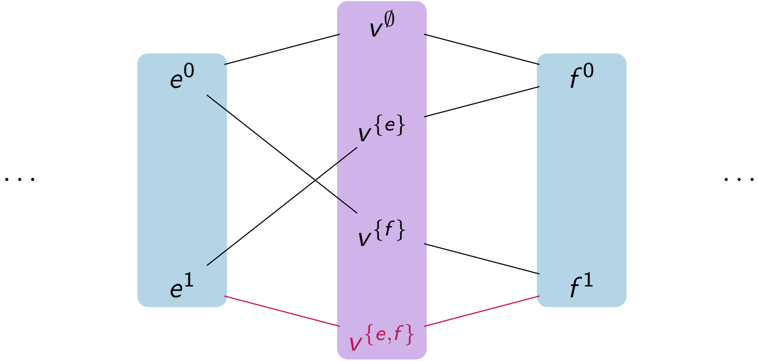

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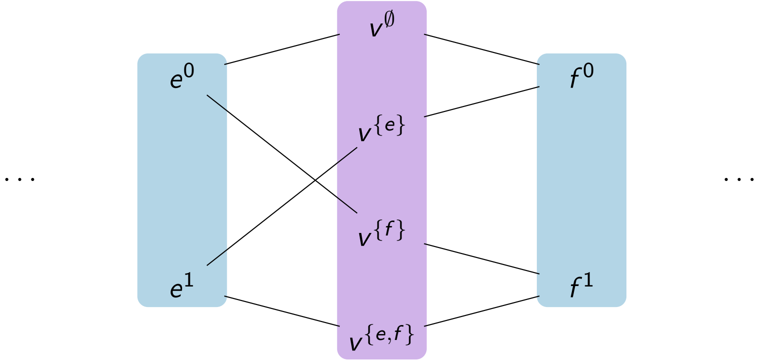

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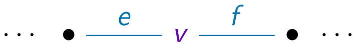


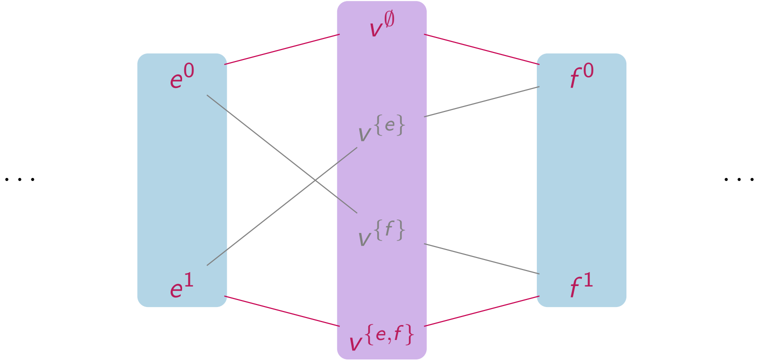


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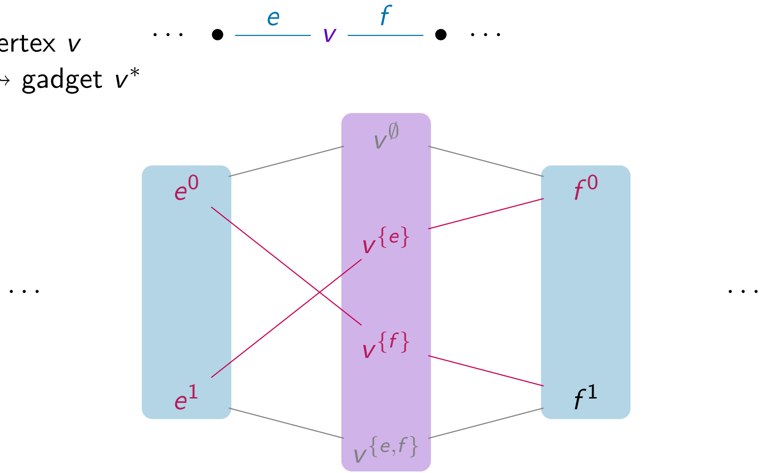
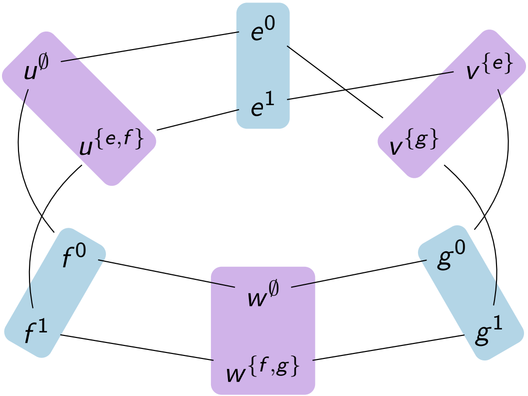



odd gadget, $v \in T$

# Cai-Fürer-Immerman graphs: example



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## Easy PTIME procedure

1. Label edge gadgets
2. Count odd vertex gadgets

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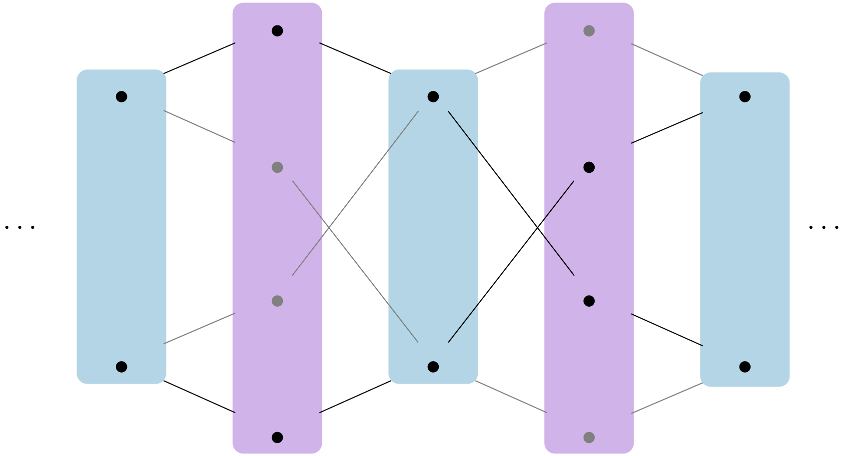

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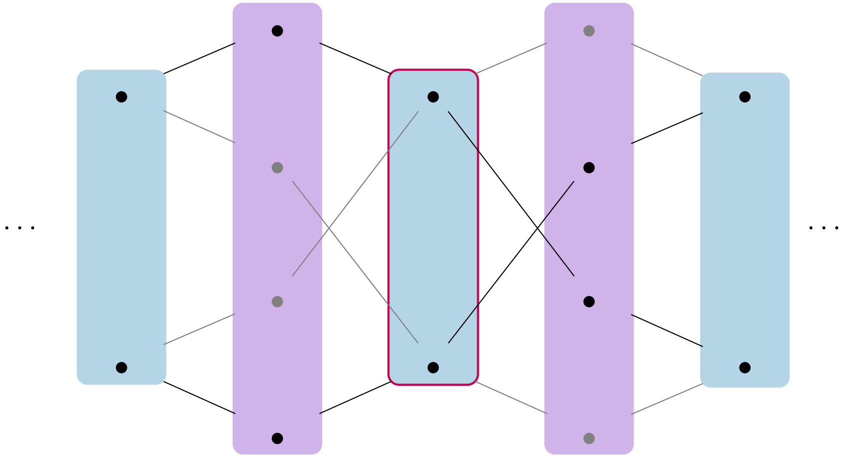

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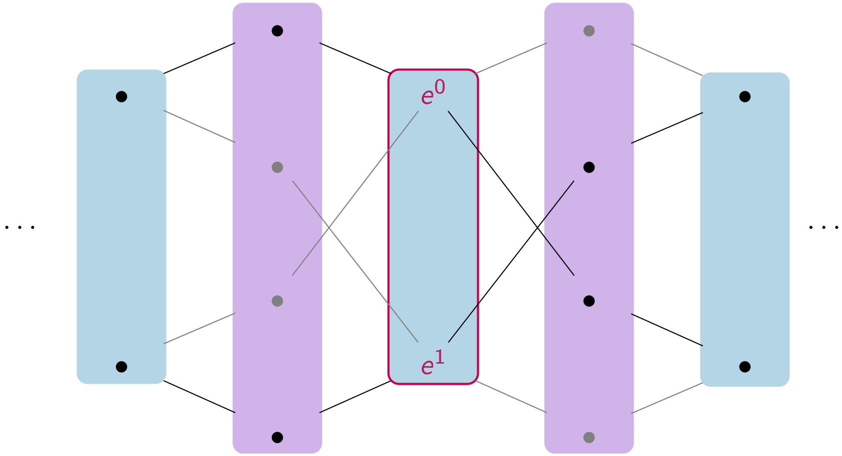

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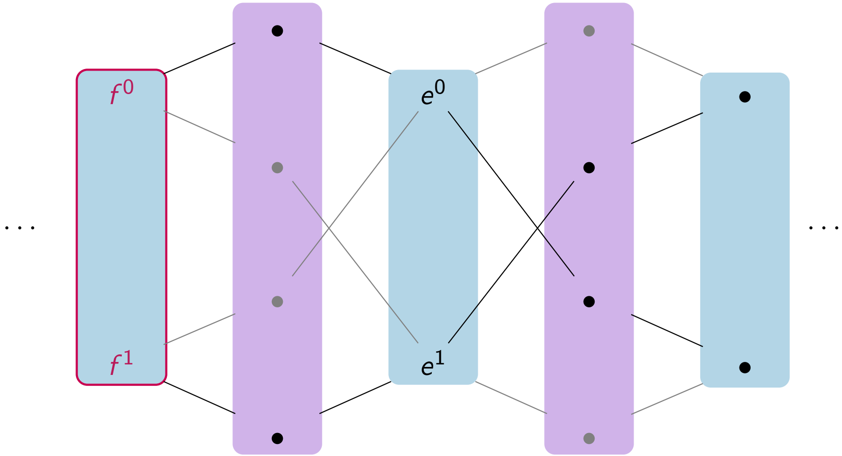

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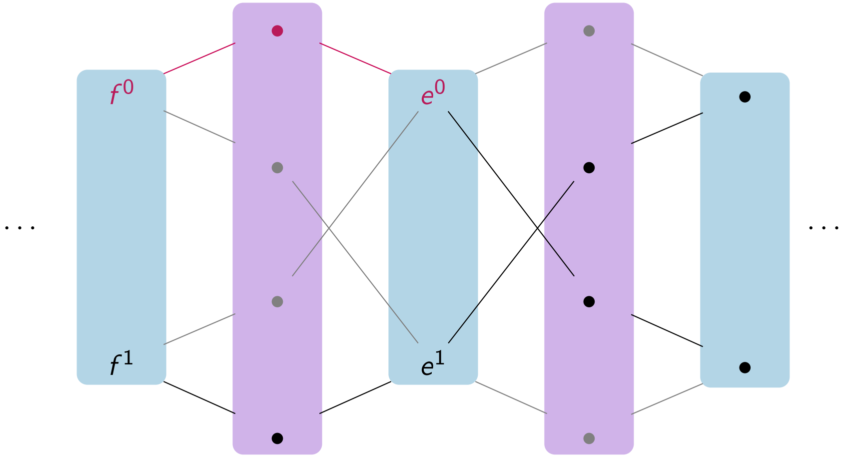

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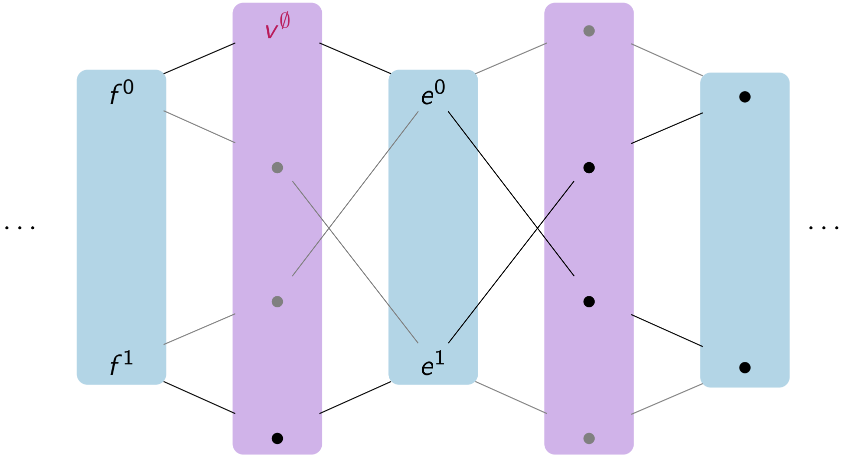

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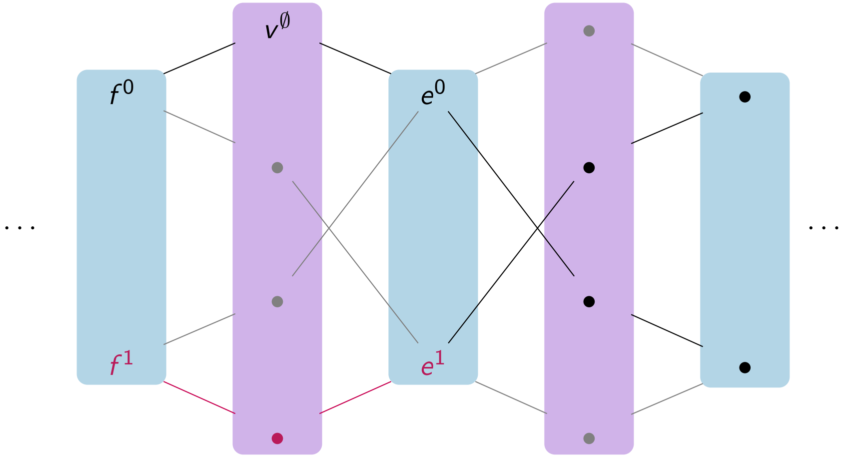

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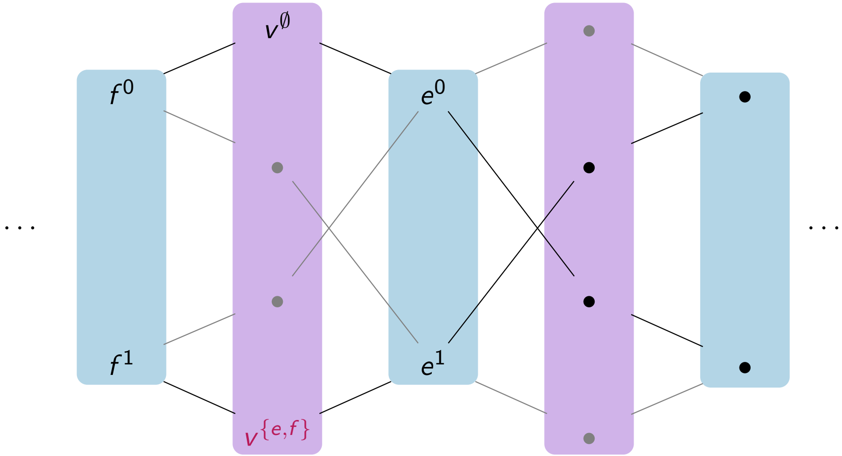

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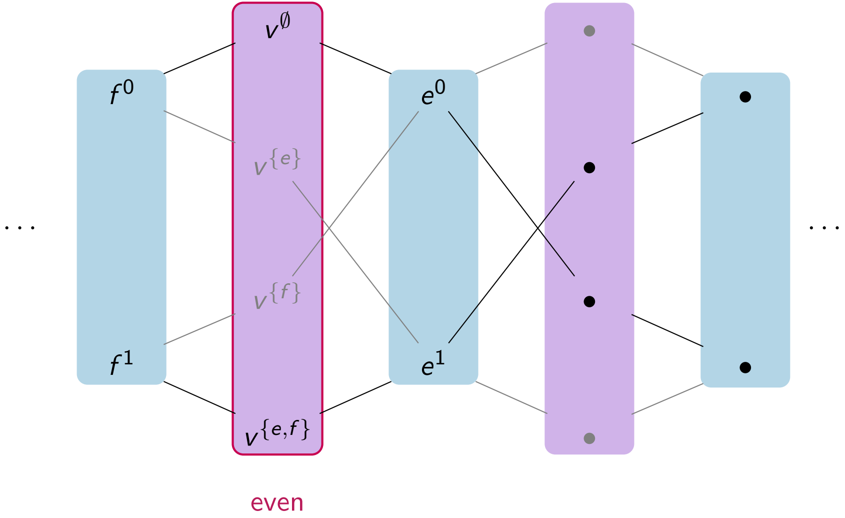

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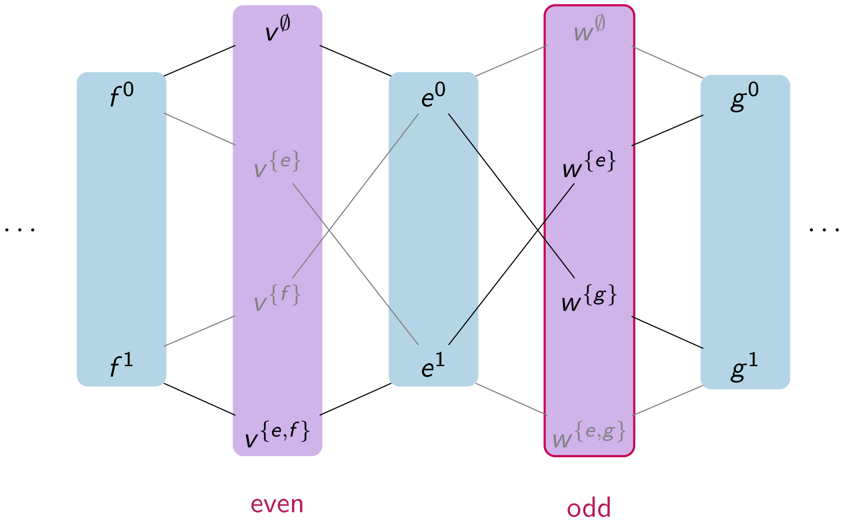

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## CPT procedure (Dawar, Richerby, Rossman)

1. Construct super-symmetric objects
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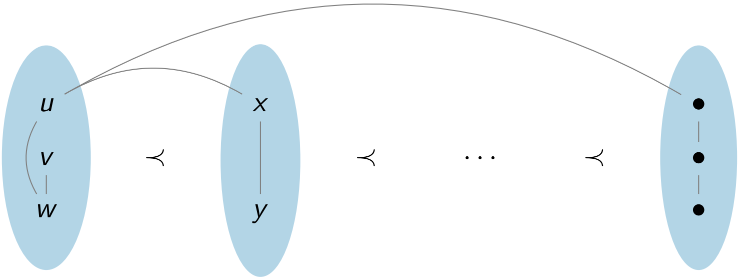
# Graphs with colour classes of logarithmic size



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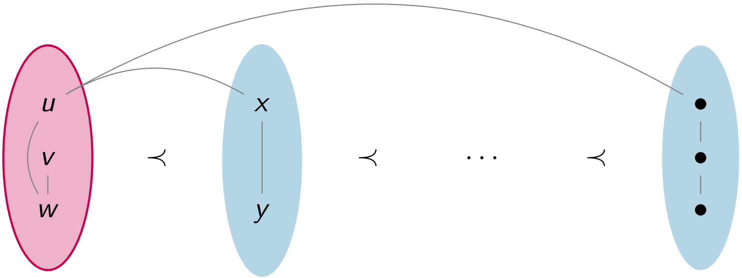


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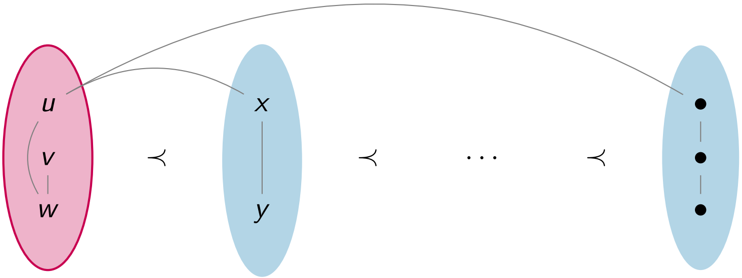


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Construct $\mathcal{O}(2^{|C|})$ many sets

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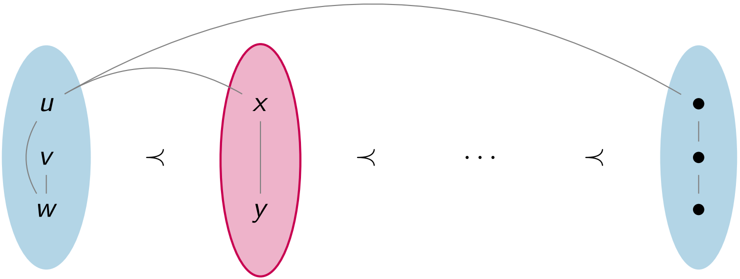


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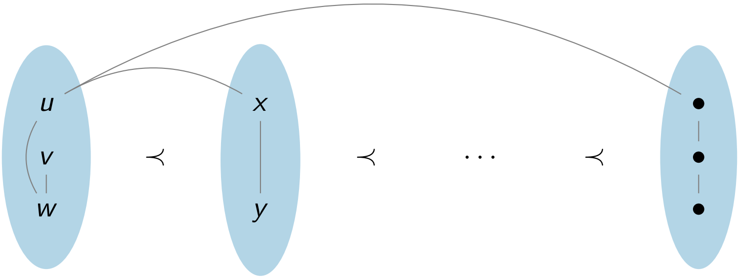


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$\mu_V$

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**Theorem (Dawar, Richerby, Rossman)**

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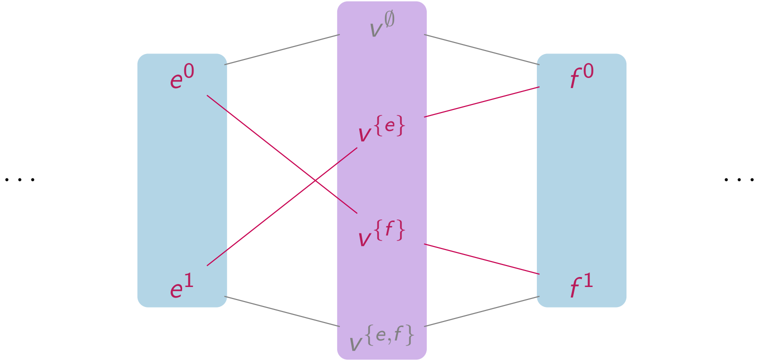
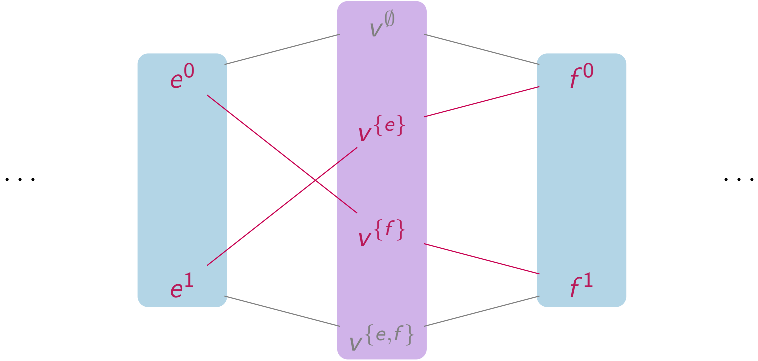


# Graphs with linear $\Delta(G)$: Keeping the rank small

- Access to all subsets of $V$
- Intuition: "Ordered" objects need nesting



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$v_1, \ldots, v_k$ vertices of $\mathcal{K}_n$.

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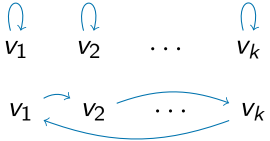


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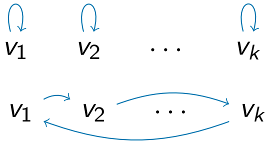


$y = (v_1, \ldots, v_k)$


$\rightsquigarrow$ strong support

# Results

**Theorem**

*The CFI query over graphs with logarithmic colour classes is* $\mathrm{CPT}$*-definable.* ✓

**Theorem**

*The CFI query over graphs with* $\Delta(G)$ *linear is* $\mathrm{CPT}$*-definable using only sets of bounded rank.* ✓

**Theorem**

*The CFI query over complete graphs is not* $\mathrm{CPT}$*-definable without using set-like objects.* ✓

**Corollary**

$\approx$ *-free* $\mathrm{PIL} \not\equiv \mathrm{CPT}[\mathrm{rk} \leq k]$✓