Mathematical Foundations for Denotational Semantics for Combining Probability and Nondeterminism over Stably Compact Spaces

Diplomarbeit

Ben S. Cohen

ENSIMAG (INP Grenoble)

TU Darmstadt

Université Joseph Fourier

Supervised by Klaus Keimel (TU Darmstadt, Fachbereich Mathematik, AG 1)
Dominique Duval (Université Joseph Fourier, IMAG, LMC)

Warranty

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Contents

Introduction

Goal and scientific context. It is the aim of the present work to generalise the mathematical theory used for the denotational semantics of certain programming languages. More precisely, we deal with languages allowing for both probabilistic and non-deterministic choice. Theories have already been elaborated by different researchers for finite state spaces and for the case that the state space is a continuous domain. We now want to generalise to stably compact state spaces, which comprise a large class of domains, as well as all compact Hausdorff spaces.

Concerning the case of finite state spaces, McIver and Morgan, formerly from the Programming Research Group in Oxford, have done considerable work, which is collected in [MM04]. There, they present a semantics together with a verification calculus in the spirit of Dijkstra's predicate transformer logic.

In the case of continuous domains as state spaces, a solid theory has been developed by Tix, Keimel, Plotkin in [TKP05]—based on previous work by Tix ([Tix99])—and independently by Mislove (see [MOW04]).

Motivation. Before diving into the subject, I would like to give a brief motivation. Two questions arise naturally: Why would we want to have probabilistic and nondeterministic features in a programming language? And: Why would we want other state spaces than continuous domains?

As to the first question, there are obvious applications for probabilistic features, and less obvious ones for non-deterministic choice. For instance, any of the popular prime number tests used for cryptography systems are based on some random choice of a natural number. Other examples are graph related problems, or sorting algorithms like quick-sort. The latter often uses an initial random permutation to decrease the probability of the worst case arising. Non-determinism on the other hand is used in different ways: to model programs that interact with a larger system, like the operating system or a user; to describe abstraction or under-specification, where the programmer wants to leave an implementation detail open; and finally, to model parallelism as the possible interleavings of concurring program bits.

Now, why use stably compact spaces instead of continuous domains? Indeed, the question is justified, since dcpo's and continuous domains have been studied since the 1970s, and are extensively used to model the denotational semantics of programming languages without probabilistic or non-deterministic features. The benefit of generalising this to certain stably compact spaces is threefold: First, it further extends the range of possible state spaces, i.e. of types to use. In fact, Martín Escardó is currently developing a functional language that has certain compact spaces as built-in data types, and that allows for certain of our cone constructions as type constructions. Secondly, the whole theory gets clarified, and the separation theorems in Section 3 may be of use to anyone working with topological cones. Finally, a direct link is established to the Dutch School, which uses complete metric spaces instead of domains for semantics. This may result in fruitful exchanges.

Organisation of the material. In Section 1, we begin by refreshing the basic concepts of domain theory that we need. We introduce the notion of a cone in its different forms, as well as some topological and categorical notions. In Section 2, we introduce the extended probabilistic powerdomain, a construction conceived to model probabilistic choice. In Section 3, we prepare several separation theorems for cones, needed in the sequel. In Section 4 and 5, we present two constructions, the convex Smyth powercone and the convex Hoare powercone, to model nondeterministic choice of programs. We finish by giving a brief summary and an overview of open questions.

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1 The Basics

The starting point of our investigation is *domain theory*. The basic concepts we need involve mainly order and topology. As a reference for domain theory, see [GHK⁺03]. The interplay of topology and order has been motivated by Smyth in [Smy83], that of topology and computability by Vickers in [Vic89].

1.1 Order

We begin by briefly recalling the concepts related to partial orders.

Definition 1.1. Consider a set X equipped with a binary relation $\sqsubseteq$ as well as the following conditions:

(R)	$(\forall x \in X)$	$x \sqsubseteq x$	(*reflexivity*)
(T)	$(\forall x, y, z \in X)$	$x \sqsubseteq y$ and $y \sqsubseteq z$ imply $x \sqsubseteq z$	(*transitivity*)
(AS)	$(\forall x, y \in X)$	$x \sqsubseteq y$ and $y \sqsubseteq x$ imply $x = y$	(*antisymmetry*)

A binary relation $\sqsubseteq$ on a set X will be called a *preorder*, if it satisfies the conditions of reflexivity (R) and transitivity (T). If, in addition, it is antisymmetric (AS), it will be called a *partial order*. The pair $(X, \sqsubseteq)$ will then be called a *preordered* resp. *partially ordered set* (*poset* for short).

Definition 1.2. A function $f\colon X \to Y$ between preordered sets $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ will be called *monotone* or *order preserving*, if we have the property $x \sqsubseteq_X y \implies f(x) \sqsubseteq_Y f(y)$ for all $x, y \in X$.

Definition 1.3. Let $(X, \sqsubseteq)$ be a preordered set. Then for any subset $D \subseteq X$ we fix the following nomenclature and notation:

- We say that D is *upper directed* or just *directed*, if it is nonempty and for any $x, y \in D$, there is an upper bound $z \in D$.
- The set $\uparrow D$ denotes the set of all elements above some elements in D:

  $$\uparrow D := \left\{ x \in X \;\middle|\; (\exists d \in D)\ d \sqsubseteq x \right\}.$$

  If D contains only one element x, we will write $\uparrow x$ instead of $\uparrow\{x\}$ for brevity. Analogously, we define $\downarrow D$ and $\downarrow x$ to denote all elements below elements in D, resp. x.
- D is called an *upper set*, if $D = \uparrow D$, and a *lower set*, if $D = \downarrow D$.

Remark. We obviously have $\uparrow D = \bigcup_{d \in D} \uparrow d$, and analogously for $\downarrow D$. Also note that, by reflexivity of $\sqsubseteq$, we have $D \subseteq \uparrow D$ and $D \subseteq \downarrow D$. The property of being an upper (lower) set is preserved under arbitrary unions and intersections.

Definition 1.4. A poset X is called a ***d**irected **c**omplete **po**set*, or just dcpo, if every directed subset $D \subseteq X$ has a supremum (i.e. a least upper bound) in X. We write $\bigsqcup^{\uparrow} D$ for the supremum of a directed set D.

Remark. For brevity, we will not always explicitly state that a subset D is directed. By writing $\bigsqcup^{\uparrow} D$ we will mean: the subset D is directed, and its supremum is $\bigsqcup^{\uparrow} D$.

Definition 1.5. An *inf semilattice* is a poset X in which any two elements $x, y \in X$ have an infimum, denoted by $x \wedge y$. Equivalently, an inf semilattice is a poset in which every finite nonempty subset has an infimum.

A *sup semilattice* is a poset X in which any two elements $x, y \in X$ have a supremum, denoted by $x \vee y$. Equivalently, a sup semilattice is a poset in which every finite nonempty subset has a supremum.

Where it is clear whether we mean a sup or an inf semilattice, we will just use the term *semilattice*.

A *complete semilattice* is a poset where every nonempty subset has an infimum and every directed subset has a supremum.

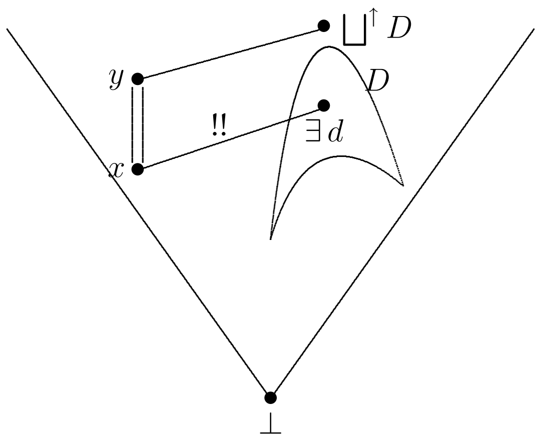
Definition 1.6. For a dcpo $(X, \sqsubseteq)$, we define a binary relation $\ll$. The relation $x \ll y$ holds if whenever a directed supremum $\bigsqcup^{\uparrow} D$ lies above y, there is some element in D that is already above x:

$$x \ll y \quad :\Longleftrightarrow \quad y \sqsubseteq \bigsqcup\nolimits^{\uparrow} D \Longrightarrow \big(\exists d \in D\big)\ x \sqsubseteq d.$$

The relation $\ll$ is called *way-below relation* or *'order' of approximation*. We will write $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} x := \{y \in X \mid y \ll x\}$ and $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} D := \{y \in X \mid (\exists d \in D)\ y \ll d\}$, and accordingly for $\mathrel{\rotatebox[origin=c]{90}{$\twoheadrightarrow$}} x$ and $\mathrel{\rotatebox[origin=c]{90}{$\twoheadrightarrow$}} D$.

Remark. Note that $\ll$ is in general *not* an order relation, because it may lack reflexivity.

The usual picture to help visualise the way-below relation is the following:



Definition 1.7. A dcpo (resp. semilattice) X is called a *continuous domain* (resp. *continuous semilattice*), if every element is the directed supremum of the elements way below it:

$$\big(\forall x \in X\big)\ x = \bigsqcup\nolimits^{\uparrow} \mathord{\downarrow\!\!\downarrow} x.$$

Remark. I define the notion of a continuous domain without saying what a domain should be. The reason is that the word 'domain' has many different meanings in different publications: sometimes it stands for dcpo, sometimes for continuous domain (e.g. in [GHK+03]), sometimes even more conditions are required. By using the term *continuous domain*, I thus try to avoid confusion.

1.2 Topology

We assume basic knowledge of topology, such as the concept of a topological space, open and closed sets, neighbourhoods, bases and subbases of a topology, nets, and continuous functions. The purpose of this subsection is not to give a primer on topology, but to refresh some concepts that the reader may have known but forgotten, to fix the notation at certain points, and most importantly to establish the essential links between topology and order that are commonly used in domain theory.

We begin with a characterisation of continuity that will prove useful in Section 5:

Proposition 1.8. *Let $f\colon X \to Y$ be a function between topological spaces. Then the following assertions*

(i) *The map f is continuous.*

(ii) *We have* $f(\overline{E}) \subseteq \overline{f(E)}$ *for any subset* $E \subseteq X$.

(iii) *We have* $\overline{f(\overline{E})} = \overline{f(E)}$ *for any subset* $E \subseteq X$.

are equivalent.

Proof.

(i) ⇒ (ii): Assume f is continuous, let $E \subseteq X$ be any subset. The closure operator $\overline{\cdot}$ is monotone and order preserving. We get

$$E \subseteq f^{-1}\Big(f(E)\Big) \quad \text{whence} \quad \overline{E} \subseteq \overline{f^{-1}\Big(f(E)\Big)}.$$

By continuity of f, the preimage of a closed set is closed, so we get:

$$f^{-1}\Big(f(E)\Big) \subseteq \underbrace{f^{-1}\Big(\overline{f(E)}\Big)}_{\text{closed}} \quad \text{whence} \quad \overline{f^{-1}\Big(f(E)\Big)} \subseteq f^{-1}\Big(\overline{f(E)}\Big).$$

These two inclusions combine to $\overline{E} \subseteq f^{-1}\big(\overline{f(E)}\big)$, which is equivalent to $f(\overline{E}) \subseteq \overline{f(E)}$, i.e. (ii).

(ii) ⇒ (i): Assume $f(\overline{E}) \subseteq \overline{f(E)}$ holds for any subset $E \subseteq X$. Choose a closed subset $C \subseteq Y$. Set $E := f^{-1}(C)$. We have:

$$\begin{array}{rrcll} & f(\overline{E}) & \subseteq & \overline{f(E)} & \text{by assumption} \\ \iff & \overline{E} & \subseteq & f^{-1}\Big(\overline{f(E)}\Big) & \text{by standard set theory} \\ \iff & \overline{f^{-1}(C)} & \subseteq & f^{-1}(C) & \text{since } E = f^{-1}(C). \end{array}$$

Since the closure operator is monotone, we also have the reverse inclusion $f^{-1}(C) \subseteq \overline{f^{-1}(C)}$, whence we have equality. Thus, the preimage of the closed set C is again closed, hence f is continuous.

(ii) ⇒ (iii): By assumption, $f(\overline{E})$ is contained in $\overline{f(E)}$. The latter is a closed set, hence it also contains the closure of $f(\overline{E})$. The reverse inclusion is clear from he monotonicity of the closure operator, hence we get the equality $\overline{f(\overline{E})} = \overline{f(E)}$.

(iii) ⇒ (ii): We have $f(\overline{E}) \subseteq \overline{f(\overline{E})}$ since the closure operator is monotone. By assumption, we have $\overline{f(\overline{E})} = \overline{f(E)}$, hence we have in particular $\overline{f(\overline{E})} \subseteq \overline{f(E)}$. Together, these two inclusions give $f(\overline{E}) \subseteq \overline{f(E)}$, which was to show.

□

Using this proposition, one can prove a lemma which we will need in Section 5. We quote it from [TKP05] without explicit proof.

Lemma 1.9. ([TKP05, Lemma 1.2]) *Let X, Y and Z be topological spaces and let*

$$f\colon X \times Y \to Z$$

be separately continuous, that is, $x \mapsto f(x, y)$ is continuous on X for every $y \in Y$ and similarly for the second coordinate. Then one has

$$\overline{f\big(\overline{A} \times \overline{B}\big)} = \overline{f\big(\overline{A} \times B\big)} = \overline{f\big(A \times \overline{B}\big)} = \overline{f\big(A \times B\big)}.$$

for all subsets $A \subseteq X$ and $B \subseteq Y$.

Now we can turn to the interplay of topology and order.

Definition 1.10. Let $(X, \mathcal{O})$ be a topological space, let $\mathcal{N}(x)$ denote the neighbourhood filter of a point x in X. Then the binary relation $\sqsubseteq$ on X defined by

$$x \sqsubseteq y \quad :\Longleftrightarrow \quad \mathcal{N}(x) \subseteq \mathcal{N}(y)$$

is called the *specialisation preorder* on X.

There are many equivalent ways to define the specialisation preorder:

$$\begin{aligned}
x \sqsubseteq y \quad &:\Longleftrightarrow \quad \mathcal{N}(x) \subseteq \mathcal{N}(y) \\
&\Longleftrightarrow \quad \overline{\{x\}} \subseteq \overline{\{y\}} \\
&\Longleftrightarrow \quad x \in \overline{\{y\}} \\
&\Longleftrightarrow \quad \big(\forall U \in \mathcal{N}(x)\big)\ y \in U.
\end{aligned}$$

It is easy to check that $\sqsubseteq$ is indeed a preorder. It turns out to be an order if and only if the space satisfies the T_0 separation axiom. We will then call it *specialisation order*. The specialisation preorder is trivial—that is, corresponds to the equality on the space—if and only if the space satisfies the T_1 separation axiom. Since we are interested in order relations, all topological spaces considered in this paper will be supposed to be at least T_0. When talking about order on a topological space, we will always mean the respective specialisation order.

Here is a reminder of the T_0 and T_1 separation axioms:

Definition 1.11. A topological space $(X, \mathcal{O})$ is said to satisfy the T_0 *separation axiom*, if for any two distinct points in X, we can separate the first from the second or the second from the first by an open set:

$$x \neq y \implies \Big(\big(\exists U \in \mathcal{N}(x)\big)\ y \notin U \ \text{ or } \ \big(\exists U \in \mathcal{N}(y)\big)\ x \notin U\Big).$$

Definition 1.12. A topological space $(X.\mathcal{O})$ is said to satisfy the T_1 *separation axiom*, if for any two distinct points in X, we can separate the first from the second and the second from the first by an open set, respectively:

$$x \neq y \implies \Big(\big(\exists U \in \mathcal{N}(x)\big)\ y \notin U \ \text{ and } \ \big(\exists U \in \mathcal{N}(y)\big)\ x \notin U \Big).$$

It is important to notice and straightforward to verify that with respect to the specialisation order of a topological space, all open sets are upper sets.

Definition 1.13. A subset of a topological space is called *saturated*, if it is the intersection of open sets.

Proposition 1.14. *Let D be a saturated subset of a topological space $(X, \mathcal{O})$. Then D is equal to the intersection of all open sets containing it:*

$$D = \bigcap \big\{ \, U \in \mathcal{O} \;\big|\; D \subseteq U \, \big\}.$$

Proof. Since D is saturated, there is a collection of open sets $(U_i)_{i \in I}$ such that $D = \bigcap_{i \in I} U_i$. Since every U_i is open and contains D, the intersection of *all* open sets containing D cannot be a larger set than D. However, by definition, the intersection of all open sets containing D must contain D, so it cannot be a smaller set than D either. Hence, the two intersections represent the same set. □

Equivalently, we could characterise the saturated subsets as being exactly those which are upper with respect to the specialisation order on the space. Since open sets are upper sets, this is an easy exercise.

To any subset which is not saturated, we can form the intersection of all open sets containing it. This will be called the *saturation* of that subset. By the previous observation, the saturation of a subset D is $\uparrow D$.

Definition 1.15. On a dcpo X, the *Scott topology* $\sigma(X)$ consists of those upper sets $U = \uparrow U$ that cannot be reached by directed suprema from outside:

$$\big(\forall D \subseteq X \ \text{ directed}\big) \ \textstyle\bigsqcup^{\uparrow} D \in U \implies \big(\exists d \in D\big) \ d \in U.$$

Remark. Equivalently, the closed sets in the Scott topology are the lower sets $A = \downarrow A$ which cannot be left by directed suprema from the inside: $\big(\forall D \subseteq A \ \text{ directed}\big) \ \bigsqcup^{\uparrow} D \in A$.

Definition 1.16. A function f between dcpo's is called *Scott-continuous*, if it preserves directed suprema, i.e. if

$$f\big(\textstyle\bigsqcup^{\uparrow} D\big) = \textstyle\bigsqcup^{\uparrow} f(D)$$

for every directed set D.

Remark. Scott-continuous functions are in particular monotone. Furthermore, as the name suggests, it turns out that the Scott-continuous functions are exactly those which are continuous with respect to the Scott topologies on the dcpo's.

Definition 1.17. On a T_0 space, the *Co-compact topology* has as basis for the closed sets all compact saturated sets. The *Patch topology* is then generated by the original topology and the co-compact topology. That is, the patch topology is the smallest topology containing both the original topology and the co-compact topology.

Definition 1.18. On a T_0 space X, the *upper topology* has as basis for the closed sets the sets $\downarrow x$, for $x \in X$. Dually, the *lower topology* has as basis for the closed sets the sets $\uparrow x$, for $x \in X$.

Remark. The nomenclature may be confusing at first. Note however that the open sets in the upper topology are upper sets (being the complements of closed sets generated by $\downarrow x$), and that the open sets in the lower topology are lower sets (being the complements of closed sets generated by $\uparrow x$).

Definition 1.19. On a T_0 space, the *Lawson topology* is generated by the Scott topology and the lower topology . That is, the Lawson topology is the smallest topology containing both the Scott topology and the lower topology.

1.3 The Extended Reals

This subsection is useful in two ways: On the one hand, it provides two examples for the notions we just introduced. And on the other hand, it presents two spaces that we will extensively use in the rest of this paper, as spaces of scalars, as weights we assign to open sets, and so on. These two spaces are the *nonnegative reals*

$$\mathbb{R}_+ := \left\{ r \in \mathbb{R} \;\middle|\; r \geq 0 \right\}$$

and the *extended nonnegative reals*

$$\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}.$$

Order. Note that both of them contain zero as an element. Unless otherwise stated, we equip $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ with the usual order, which is extended in the case of $\overline{\mathbb{R}}_+$ to take care of ∞ as the largest element. Thanks to the added ∞ value, $\overline{\mathbb{R}}_+$ is a dcpo, and even a continuous domain. The way below relation on both is just the usual 'strictly less' relation $<$ (except for $0 \ll 0$).

Topology. The Scott topology is given by the sets $]r, \infty[$ with $r \geq 0$ on $\mathbb{R}_+$ and by the sets $]r, \infty]$ with $r \geq 0$ on $\overline{\mathbb{R}}_+$. A function $f\colon X \to \overline{\mathbb{R}}_+$ that is continuous w.r.t. the Scott topology on the target space $\overline{\mathbb{R}}_+$ is called *lower semicontinuous* in classical analysis. The space of lower semicontinuous functions is denoted by $\mathcal{L}(X)$.

Arithmetic. In Section 2, we will integrate lower semicontinuous functions with respect to measure-like functions. Though we will not go to great length and detail in this matter, we must define how to extend the algebraic operations on the reals:

$$\begin{array}{rclcll} x + \infty & = & \infty & = & \infty + x, & \text{for } x \in \overline{\mathbb{R}}_+ \\ x \cdot \infty & = & \infty & = & \infty \cdot x, & \text{for } x \in \overline{\mathbb{R}}_+ \backslash \{0\} \\ 0 \cdot \infty & = & 0 & = & \infty \cdot 0, & \end{array}$$

With this, addition and multiplication are Scott-continuous on $\overline{\mathbb{R}}_+$, turning it into a continuous dcpo-cone (see below in Subsection 1.4).

1.4 Cones

When modelling the interplay of nondeterministic and probabilistic choice, we will use certain convex combinations. Hence, the need for structures with addition and scalar multiplication arises.

Definition 1.20. A set C endowed with two operations, that of addition $+\colon C \times C \to C$ and that of scalar multiplication $\cdot\colon \mathbb{R}_+ \times C \to C$ is called a *cone*, if the following hold: There is a neutral element $0 \in C$ for addition turning $(C, +, 0)$ into a commutative monoid, that is, for all $a, b, c \in C$ one has:

$$\begin{array}{rcl} (a + b) + c & = & a + (b + c) \\ a + b & = & b + a \\ a + 0 & = & a. \end{array}$$

Moreover, scalar multiplication acts on this monoid as on a vector space:

For $a, b \in C$ and $r, s \in \mathbb{R}_+$, one has

$$\begin{aligned} 1 \cdot a &= a \\ 0 \cdot a &= 0 \\ (r \cdot s) \cdot a &= r \cdot (s \cdot a) \\ r \cdot (a + b) &= (r \cdot a) + (r \cdot b) \\ (r + s) \cdot a &= (r \cdot a) + (s \cdot a). \end{aligned}$$

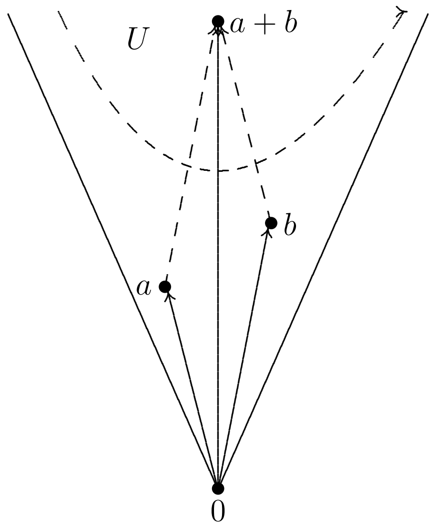
Definition 1.21. A function $f : C \to D$ between cones is called *linear* if, for all $a, b \in C$ and $r \in \mathbb{R}_+$, one has

$$\begin{aligned} f(a + b) &= f(a) + f(b) \\ f(r \cdot a) &= r \cdot f(a). \end{aligned}$$

Definition 1.22. A cone C is an *ordered cone* if it is also endowed with a partial order $\leq$ such that addition and scalar multiplication considered as maps $C \times C \to C$ and $\mathbb{R}_+ \times C \to C$, respectively, are order preserving in both variables. If the order is directed complete and if addition and scalar multiplication are Scott-continuous, then C is called a *dcpo-cone*. Thus, a dcpo-cone is at the same time a cone and a dcpo. In the case that C is a continuous domain, C is called a *continuous dcpo-cone*.

Remark. In literature, the word *d-cone* is often used instead of dcpo-cone, for brevity. In this paper, we stick to the explicit terminology, for the sake of clarity.

One can picture a cone abstractly in the following way:



The minimal element is denoted by $\mathbf{0}$, an open set U is drawn as an upper set, the dashed border should remind us that open sets have a "soft" border (since scalar multiplication by any fixed element is lower semicontinuous). Since cones have similarities with vector spaces, we often depict the points as vectors.

Definition 1.23. A *semitopological cone* is a cone C equipped with a topology such that

- addition is separately continuous (i.e. continuous in each of the two components separately) and
- scalar multiplication is jointly continuous (i.e. continuous as a function $\mathbb{R}_+ \times C \to C$)

with respect to the given topology on C and the Scott topology on $\mathbb{R}_+$.

A *topological cone* is a cone C equipped with a topology such that

- addition is jointly continuous (i.e. as a function $C \times C \to C$) and
- scalar multiplication is jointly continuous (i.e. continuous as a function $\mathbb{R}_+ \times C \to C$)

with respect to the given topology on C and the Scott topology on $\mathbb{R}_+$.

Remark. It is easy to verify that continuous functions preserve the specialisation order. Hence, every topological cone is an ordered cone with respect to its specialisation order. Note that for monotonicity, it does not matter if we check the two components of scalar multiplication separately or jointly. The same holds true for the property of preserving directed suprema, since with $\mathbb{R}_+$, one of the two factors is a continuous domain (see [TKP05, p. 14]). For the same reason, joint continuity and separate continuity are equivalent for scalar multiplication (a continuous domain has a completely distributive lattice of open sets by [GHK$^+$03, Theorem II-1.14, p.142], hence is an α-space by [Ers97, Theorem 3], hence for a function such as scalar multiplication, separate and joint continuity agree by [Ers97, Proposition 2]). Therefore, we will always only check for separate continuity in the proofs.

Now that we have introduced different cone structures, which allow for addition and scalar multiplication, it makes sense to define the following:

Definition 1.24. A subset U of a cone is called *convex*, if for each pair of points in U, the line segment connecting them is also contained in U:

$$\big(\forall x, y \in U\big) \ \big(\forall r \in [0,1]\big) \quad r \cdot x + (1-r) \cdot y \in U.$$

A topological cone is called *locally convex*, if every point has a neighbourhood basis of open convex sets.

Remark. Note that our notion of convexity is of geometrical nature and not to be confused with *order convex* which means that $a, b \in U$ and $a \leq x \leq b$ imply $x \in U$. However, upper sets are automatically order-convex, so open convex sets are also order-convex w.r.t. the specialisation order.

1.5 Stably Compact Spaces

In this subsection, we define stably compact spaces, which we will use for our constructions in Section 2 and Section 4. The class of stably compact spaces subsumes most semantic domains and has many other closure properties which are interesting for semantics (see [Keg99]). Stably compact spaces are useful if one wants to stick with T_0 spaces but enjoy properties that compact sets have in Hausdorff spaces. An example for a stably compact space is $\overline{\mathbb{R}}_+$ equipped with the Scott topology.

Definition 1.25. A subset of a topological space is called *compact*, if it satisfies the Heine-Borel property, i.e. for every open cover of the subset, there is a finite subcover. The space itself is called *locally compact* if, for every point, every neighbourhood contains a compact neighbourhood.

Remark. Since the Heine-Borel property is about open coverings and open sets are upper sets, a set is compact if and only if its saturation is. Thus, we will often consider compact saturated sets instead of just compact ones.

Definition 1.26. Given a collection $\mathcal{F} \subseteq \mathfrak{P}(X)$ of subsets of a topological space $(X, \mathcal{O})$, consider the following assertions:

(F0) $\mathcal{F} \neq \varnothing$

(F1) $\varnothing \notin \mathcal{F}$

(F2) $F_1, F_2 \in \mathcal{F} \Longrightarrow \big(\exists F_3 \in \mathcal{F}\big)\ F_3 \subseteq F_1 \cap F_2$

(F3) $F \in \mathcal{F}$ and $F \subseteq G \subseteq X \Longrightarrow G \in \mathcal{F}$.

Then $\mathcal{F}$ is called a *filter base*, if it satisfies (F0)–(F2). If, in addition, it satisfies (F3), then it is called a *filter*.

Remark. Note that (F2) and (F3) imply that a filter is closed under finite intersection.

Definition 1.27. A topological space is called *coherent*, if the intersection of any two compact saturated subsets is again compact.

Definition 1.28. We call a topological space $(X, \mathcal{O})$ *well-filtered*, if the following holds: For any filter base $\mathcal{F}$ of compact saturated sets, if $\bigcap \mathcal{F}$ is contained in an open set U, then there is $F \in \mathcal{F}$ such that F is already contained in U.

Definition 1.29. A topological space is called *stably compact*, if it is compact, locally compact, coherent and well-filtered.

Remark. Stably compact spaces are often defined to be *sober* instead of well-filtered. A space X is *sober*, if for every irreducible closed set C there is a unique $x \in X$ with $\overline{\{x\}} = C$. A nonempty set C is irreducible, if $C \subseteq D \cup E$ implies $C \subseteq D$ or $C \subseteq E$.

The presence of local compactness makes sober and well-filtered equivalent properties (see [GHK⁺03, Theorem II-1.21, p.147]), so our definition is just a reformulation of the standard one.

1.6 Categories

We assume knowledge of the basic concepts in category theory, such as categories, morphisms and functors. We briefly recall the definition of a natural transformation and a monad, that we will need in Section 2.

Definition 1.30. Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\tau\colon F \dot{\to} G$ assigns to every object A in $\mathcal{C}$ a morphism $\tau_A\colon F(A) \to G(A)$, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[d, "f"'] & F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
B & F(B) \arrow[r, "\tau_B"'] & G(B)
\end{tikzcd}
$$

So natural transformations map functors to functors. The composition of two natural transformations τ and σ is given by $(\tau \circ \sigma)_A = \tau_A \circ \sigma_A$, for every object A.

Note that the collection of functors between two categories is again a category, with the natural transformations as morphisms.

Definition 1.31. A *monad* on a category $\mathcal{C}$ is given by a triple (T, η, μ) with $T\colon \mathcal{C} \to \mathcal{C}$ as well as $\eta\colon \mathsf{Id}_{\mathcal{C}} \to T$ and $\mu\colon T^2 \to T$ such that the following equalities hold:

$$\mu \circ \eta_T = \mathsf{Id}_T = \mu \circ T\eta \qquad \text{and} \qquad \mu \circ T\mu = \mu \circ \mu_T.$$

That is, the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Then η and μ are called the *unit* and the *multiplication* of the monad, respectively.

Definition 1.32. An *algebra* of a monad (T, η, μ) on a category $\mathcal{C}$ is given by an object A in $\mathcal{C}$ together with a morphism $\alpha\colon T(A) \to A$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^3(A) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_{T(A)}"'] & T^2(A) \arrow[d, "\alpha"] \\ T^2(A) \arrow[r, "\alpha"'] & T(A) \end{tikzcd}$$

2 The Extended Probabilistic Powerdomain $\mathcal{V}(X)$

We want to give the denotational semantics of programs using probabilistic choice. More concretely, we want to allow the program to choose between a finite number of program sections, where each choice has a given probability to occur. Since finite choice can be obtained by finite use of binary choice, this is equivalent to having an instruction like this:

$$P_1 \oplus_p P_2$$

The intended meaning of such a construct is: The program P_1 is executed with probability p, and the program P_2 is executed with probability $1 - p$, where $0 < p < 1$ is a real number.

If a program is allowed to make such choices, the outcome of the program may be different for different executions. The behaviour of the program is then no longer uniquely determined, but rather given by some distribution on the state space.

This approach has been studied since the early 1980s first by Saheb-Djahromi, then by Plotkin and Jones, who have established the notion of a *probabilistic powerdomain* ([Sah80, Plo82, JP89, Jon90]), which has later been extended to the *extended probabilistic powerdomain*. See [TKP05, Introduction] for a brief overview of these approaches. These constructions are spaces of *valuations*, which are the topological flavour of distributions.

In this section, we will generalise parts of [TKP05, Section 2.2] on the extended probabilistic powerdomain by considering stably compact spaces instead of continuous domains as state spaces.

2.1 The Definition

Definition 2.1. On a topological space $(X, \mathcal{O})$, a function $\mu\colon \mathcal{O} \to \overline{\mathbb{R}}_+$ that assigns to every open set a nonnegative extended real number is called a *valuation*, if for all $U, V \in \mathcal{O}$ it satisfies the following conditions:

$$\begin{array}{rcll} \mu(\varnothing) & = & 0 & \text{(strictness)} \\ U \subseteq V & \Rightarrow & \mu(U) \leq \mu(V) & \text{(monotonicity)} \\ \mu(U) + \mu(V) & = & \mu(U \cup V) + \mu(U \cap V) & \text{(modularity)} \end{array}$$

If, in addition, it preserves directed suprema, i.e. if for any directed family of open sets $(U_i)_{i \in I}$ it satisfies

$$\mu\Big(\bigcup_{i\in I}^{\uparrow} U_i\Big) = \bigsqcup_{i\in I}^{\uparrow} \mu(U_i),$$

then it is called a *continuous valuation*.

Remark. Since the collection of open sets on a topological space is ordered by inclusion, we can indeed talk about a directed family of open sets.

Examples of valuations include *point valuations*, defined by

$$\eta_x \colon \mathcal{O} \to \overline{\mathbb{R}}_+ \qquad \text{with} \qquad \eta_x(U) := \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

and *simple valuations*, which are finite linear combinations of point valuations with nonnegative real coefficients:

$$\sum_{i=1}^{n} r_i \eta_{x_i}.$$

The centre of interest of this section is the space of all continuous valuations of some topological space:

Definition 2.2. The collection of all continuous valuations on a topological space X is called the *extended probabilistic powerdomain* on X and denoted by $\mathcal{V}(X)$.

2.2 The Topological Cone $\mathcal{V}(X)$

We will explore $\mathcal{V}(X)$ and equip it with a richer structure. First, we define a pointwise addition and scalar multiplication by setting

$$(\mu + \nu)(U) := \mu(U) + \nu(U) \qquad \text{and} \qquad (r \cdot \mu)(U) := r \cdot \mu(U).$$

Furthermore, we define an order on $\mathcal{V}(X)$ by setting

$$\mu \leq \nu \quad :\Longleftrightarrow \quad \mu(U) \leq \nu(U) \text{ for all open sets } U.$$

This turns $\mathcal{V}(X)$ into an ordered cone.

Since valuations can be considered to be a topological variant of measures or probability distributions, it is natural to ask whether they can be used for integration. Indeed, for any topological space X, the integral of a lower semicontinuous function $f \colon X \to \overline{\mathbb{R}}_+$ with respect to a continuous valuation μ can be defined. If we define the auxiliary function $f_\mu \colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ by

$$f_\mu(r) := \mu\Big(f^{-1}\big(]r, +\infty]\big)\Big),$$

then we can define the integral via an improper Riemann integral:

$$\int f \,\mathrm{d}\mu := \int_0^\infty f_\mu(r)\,\mathrm{d}r.$$

It turns out that the integral is linear in its two components, which suggests the scalar product notation $\langle \mu, f \rangle := \int f \, \mathrm{d}\mu$. Details on the integral can be found in [Tix95, Section 4].

Definition 2.3. Consider the extended probabilistic powerdomain $\mathcal{V}(X)$ over a topological space X. We define the following topologies on $\mathcal{V}(X)$:

- The *weak topology* is the coarsest topology that renders continuous the maps $\mu \mapsto \mu(U) \colon \mathcal{V}(X) \to \overline{\mathbb{R}}_+$ for all open subsets U.
- The *product topology* is the coarsest topology that renders continuous the maps $\langle \cdot, f \rangle \colon \mathcal{V}(X) \to \overline{\mathbb{R}}_+$ for all lower semicontinuous functions $f \in \mathcal{L}(X)$.

It turns out that these two topologies agree, and that with them, $\mathcal{V}(X)$ is a topological cone (see [Tix95, Lemma 4.9, Satz 4.6]). The space $\mathcal{V}(X)$ with the weak topology has even more useful properties: If we start with a stably compact space X, then $\mathcal{V}(X)$ with the weak topology is stably compact, too (see [TKP05, Theorem 2.10(c)]). Furthermore, $\mathcal{V}(X)$ is *always* locally convex:

Proposition 2.4. *If X is a topological space, then $\mathcal{V}(X)$ with the weak topology is locally convex.*

Proof. The weak topology is equal to the product topology on $\mathcal{V}(X)$, which is the coarsest topology such that the maps $\langle \cdot, f \rangle \colon \mathcal{V}(X) \to \overline{\mathbb{R}}_+$ are continuous for all lower semicontinuous functions $f \colon X \to \overline{\mathbb{R}}_+$. The sets $\langle \cdot, f \rangle^{-1}(]r, +\infty])$ are a basis of this topology. Since $\langle \cdot, f \rangle$ is linear and the open sets $]r, +\infty]$ in $\overline{\mathbb{R}}_+$ are convex, so are the basic open sets of the product topology. Now, this is just the definition of being locally convex. $\square$

2.3 The Monad $(\mathcal{V}, \eta, \mu)$

In this section, we will consider the category **StCp** of stably compact spaces and continuous maps. We begin by showing that $\mathcal{V}$ can be extended to a functor. Indeed, given a continuous function $f \colon X \to Y$ between stably compact spaces, we can define

$$\mathcal{V}(f) \colon \mathcal{V}(X) \to \mathcal{V}(Y) \qquad \text{by} \qquad \mu \mapsto \mu \circ f^{-1},$$

where f^{-1} denotes the preimage of f. Then we can show that $\mathcal{V}$ is a functor from the category **StCp** of stably compact spaces into itself (and even into the category **StCpCvCone** of stably compact locally convex topological cones, by the above):

Proposition 2.5. *The extended probabilistic powerdomain operator $\mathcal{V}$ is an endofunctor of the category* **StCp***.*

Proof. We have already seen that $\mathcal{V}$ maps the objects of **StCp** to objects of **StCp**. What we still have to check is that it maps the morphisms right. So let $f\colon X \to Y$ be a continuous function, hence a morphism in **StCp**. It is straightforward to verify that $\mathcal{V}(f)$ is well-defined and linear, using that the preimage f^{-1} preserves arbitrary union and finite intersection, as well as the pointwise definition of of $+$ and $\cdot$ on $\mathcal{V}(X)$. What remains to show is that $\mathcal{V}(f)$ is continuous w.r.t the weak topologies on $\mathcal{V}(X)$ and $\mathcal{V}(Y)$, respectively.

To this end, let $V = \left\{ \nu \in \mathcal{V}(Y) \;\middle|\; \nu(U) > r \right\}$ be a subbasic open set in $\mathcal{V}(Y)$. Then we have

$$\begin{aligned}
\mathcal{V}(f)^{-1}(V) &= \left\{ \mu \in \mathcal{V}(X) \;\middle|\; \mathcal{V}(f)(\mu) \in V \right\} \\
&= \left\{ \mu \in \mathcal{V}(X) \;\middle|\; \mu \circ f^{-1} \in V \right\} \\
&= \left\{ \mu \in \mathcal{V}(X) \;\middle|\; \left(\mu \circ f^{-1}\right)(U) > r \right\} \\
&= \left\{ \mu \in \mathcal{V}(X) \;\middle|\; \mu\left(f^{-1}(U)\right) > r \right\},
\end{aligned}$$

which is basic open in $\mathcal{V}(X)$, since f is continuous, and hence $f^{-1}(U)$ is open. □

Remark. The images of both objects and morphisms under $\mathcal{V}$ lie in the category **StCpCone** of stably compact locally convex topological cones with linear continuous functions.

Knowing that $\mathcal{V}$ is a functor, we can define suitable η and μ such that $(\mathcal{V}, \eta, \mu)$ becomes a monad in the category **StCp** of stably compact spaces. Why is this interesting? Well, if we can determine the algebras of this monad, then we might derive a universal property for the extended probabilistic powerdomain. More precisely: Suppose that to any object C in **StCpCvCone** there is a unique morphism γ turning (C, γ) into an algebra of this monad. Then for a given object C, we would have the following commuting diagram

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}"'] & \mathcal{V}(C) \arrow[d, "\gamma"] \\
& C
\end{tikzcd}$$

and for any other object D and any morphism $f\colon C \to D$, we would have a unique morphism $\hat{f}\colon \mathcal{V}(C) \to D$, given by $f \circ \gamma$:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[ddr, "f"'] & \mathcal{V}(C) \arrow[d, "\gamma"'] \arrow[dd, dashed, "\hat{f}"] \\
& C \arrow[d, "f"'] \\
& D
\end{tikzcd}$$

This universal property is important for our denotational semantics: Suppose we have probabilistic but no non-deterministic features in our programming language. A program P on the state space X is interpreted by a function $[\![P]\!]\colon X \to \mathcal{V}(X)$. Now, for two such programs P_1 and P_2, we can form the sequential composition $P_1\,;P_2$, but we cannot interpret it by concatenating the interpreting functions for P_1 and P_2—for simple type checking reasons. But if we have the universal property described above, we can find a unique extension $\widehat{[\![P_2]\!]}$ to $[\![P_2]\!]$ and form $\widehat{[\![P_2]\!]} \circ [\![P_1]\!]$ to interpret $P_1\,;P_2$.

As we will see, we can find *some* suitable extension function without knowing the algebras of our monad, just using the monad properties. So let us determine the monad of $\mathbf{StCp}$ involving $\mathcal{V}$. This for, let $\eta\colon \mathsf{Id}_{\mathbf{StCp}} \dot{\to} \mathcal{V}$ be defined by $\eta_C\colon C \to \mathcal{V}(C)$ with $x \mapsto \eta_x$ for every C in $\mathbf{StCp}$. In order to define $\mu\colon \mathcal{V}^2 \dot{\to} \mathcal{V}$, first consider for every open set U the evaluation map

$$\varepsilon_U\colon \mathcal{V}(C) \to \overline{\mathbb{R}}_+ \qquad \text{with} \qquad \xi \mapsto \xi(U).$$

All such evaluation maps are continuous w.r.t. the weak topology on $\mathcal{V}(C)$, by definition of the latter. Now, for every $\nu \in \mathcal{V}^2(C)$, define a valuation $\mu_C(\nu)\colon \mathcal{O}(C) \to \overline{\mathbb{R}}_+$ on C by setting

$$\mu_C(\nu)(U) := \int_{\xi \in \mathcal{V}(C)} \xi(U)\,\mathrm{d}\nu = \int_{\xi \in \mathcal{V}(C)} \varepsilon_U(\xi)\,\mathrm{d}\nu.$$

Now, let us verify that these are indeed natural transformations, and exhibit some of their properties:

Proposition 2.6. *Let C be a stably compact space. Then we have:*

(a) *The map η_C is injective and continuous, even an embedding.*

(b) *The map $\eta\colon \mathsf{Id}_{\mathbf{StCp}} \dot{\to} \mathcal{V}$ is a natural transformation.*

(c) *For all $\nu \in \mathcal{V}^2(C)$, the map $\mu_C(\nu)$ is a continuous valuation on C.*

(d) *The map $\mu\colon \mathcal{V}^2 \dot{\to} \mathcal{V}$ is a natural transformation.*

(e) *The map $\mu_C\colon \mathcal{V}^2(C) \to \mathcal{V}(C)$ is continuous w.r.t. the weak topology on the respective spaces.*

Proof. (a): Since C is T_0, the point valuations η_x and η_y must differ at some open set, if $x \neq y$. Hence, η_C is injective. Now, for continuity let $U \subseteq \mathcal{V}(C)$ be a subbasic open set, i.e. $U = \left\{ \xi \in \mathcal{V}(C) \;\middle|\; \xi(V) > r \right\}$ for some open set $V \subseteq C$. We have

$$\begin{aligned} \eta_C^{-1}(U) &= \left\{ x \in C \;\middle|\; \eta_C(x) \in U \right\} \\ &= \left\{ x \in C \;\middle|\; \eta_x \in U \right\} \\ &= \left\{ x \in C \;\middle|\; \eta_x(V) > r \right\}, \end{aligned}$$

which is empty for $r > 1$ and equal to V if $r \leq 1$. In both cases, the preimage is an open set, hence the map is continuous. To check that η_C is an embedding, we have to verify that the topology on C is the "trace" of the weak topology on $\mathcal{V}(C)$. That is, we have to show that for $U \subseteq \mathcal{V}(C)$ which is subbasic open, the set $\{x \in C \mid \eta_x \in U\}$ is open. This is given by the continuity of η_C which we have just shown. For $r = 0$, this gives us a set $U = \{\xi \in \mathcal{V}(C) \mid \xi(V) > 0\}$ with the property $\eta_x(V) = \eta_x(C) \cap U$ for every $x \in C$ and every open set $V \subseteq C$, hence we are done.

(b): Let $f\colon C \to D$ be a morphism in $\mathbf{StCp}$. For $x \in C$ and U open in D, we have

$$\begin{aligned} (\eta_D \circ f)(x)(U) &= \eta_{f(x)}(U) \\ &= \eta_x\left(f^{-1}(U)\right) \\ &= \left(\eta_x \circ f^{-1}\right)(U) \\ &= \mathcal{V}(f)(\eta_x)(U) \\ &= (\mathcal{V}(f) \circ \eta_D)(U). \end{aligned}$$

Hence, we have $\eta_D \circ f = \mathcal{V}(f) \circ \eta_C$ and η is a natural transformation.

(c) and (d): Here, the very same proof for the case of continuous domains instead of stably compact spaces can be taken over from [Kir93, Lemma 6.3, p.52], since Kirch uses no feature specific to continuous domains in his proof.

(e): Let $U \subseteq \mathcal{V}(C)$ be subbasic open, i.e. $U = \left\{ \xi \in \mathcal{V}(C) \;\middle|\; \xi(V) > r \right\}$ for

$V \subseteq C$ open. Then we calculate

$$\begin{aligned}
\mu_C^{-1}(U) &= \{\, \nu \in \mathcal{V}^2(C) \mid \mu_C(\nu) \in U \,\} \\
&= \{\, \nu \in \mathcal{V}^2(C) \mid \mu_C(\nu)(V) > r \,\} \\
&= \Big\{\, \nu \in \mathcal{V}^2(C) \mid \int_{\xi \in \mathcal{V}(C)} \xi(U)\, \mathrm{d}\nu > r \,\Big\} \\
&= \Big\{\, \nu \in \mathcal{V}^2(C) \mid \int_{\xi \in \mathcal{V}(C)} \varepsilon_U(\xi)\, \mathrm{d}\nu > r \,\Big\},
\end{aligned}$$

which is subbasic open since the ε_U are continuous by definition of the weak topology. □

Now we have everything ready to show the monad property:

Theorem 2.7. *The triple $(\mathcal{V}, \eta, \mu)$ is a monad in the category* $\mathbf{StCp}$ *of stably compact spaces.*

Proof. The exact same verification from [Kir93, Satz 6.1, p.54], stated for the case of continuous dcpo-cones, goes through in our setting. □

Although a lot of the investigations of continuous dcpo-cones in [Kir93] apply to our new situation, in our case the algebras of the monad $(\mathcal{V}, \eta, \mu)$ remain unknown so far. But the monad properties are sufficient to prove the extension property we need for sequential composition of programs: For a given continuous function $f\colon C \to D$ between stably compact spaces, define the map $\hat{f}\colon \mathcal{V}(C) \to \mathcal{V}(D)$ by setting

$$\hat{f} := \mu_D \circ \mathcal{V}(f)$$

By the monad properties, we have $\mu \circ \mathcal{V}\eta = \mathsf{Id}_{\mathbf{StCp}}$, i.e. $\mu_D \circ \mathcal{V}(f) \circ \eta_C = f$, hence the following diagram commutes and we have found the sought-after extension $\hat{f}$ to f:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & \mathcal{V}(C) \arrow[d, dashed, "\hat{f}"] \arrow[dr, "\mathcal{V}(f)"] & \\
& \mathcal{V}(D) & \mathcal{V}\mathcal{V}(D) \arrow[l, "\mu_D"]
\end{tikzcd}$$

Note that we have shown a property which is a bit more general than what we need, since in the concrete case of program composition, we have

$C = D$. The extension $\hat{f}$ has a certain uniqueness property which we do not need to discuss in view of our applications.

To finish this section, here is an interesting special case of our property:

Proposition 2.8. *For every stably compact space X, we have the following extension property:*

For every lower semicontinuous function $f\colon X \to \overline{\mathbb{R}}_+$, there is a continuous linear function $\hat{f}\colon \mathcal{V}(X) \to \overline{\mathbb{R}}_+$ such that $\hat{f} \circ \eta_X = f$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\eta_X"] \arrow[dr, "f"'] & \mathcal{V}(X) \arrow[d, "\hat{f}"] \\ & \overline{\mathbb{R}}_+ \end{tikzcd}$$

This function is given by $\hat{f}(\mu) := \int f \,\mathrm{d}\mu$.

Proof. By a straightforward calculation, we find $\int f \,\mathrm{d}\eta_x = f(x)$. Hence

$$\left(\hat{f} \circ \eta_X\right)(x) = \hat{f}(\eta_x) = \int f \,\mathrm{d}\eta_x = f(x),$$

so the diagram commutes. By the definition of the weak topology, the function $\hat{f}$ is continuous. Finally, $\hat{f} = \langle \cdot, f \rangle$ is linear for every topological space X (see [Tix95, Satz 4.4]). □

3 Hahn-Banach Type Theorems

In this section, we will prove separation theorems for locally convex topological cones, that we will need in Section 4. Most of it has already been proved for the case of continuous dcpo-cones in [TKP05]. Klaus Keimel and Gordon Plotkin had suggested generalisations to topological cones. It was my task to write down the proofs for these generalisations. I have examined the existing proofs to verify which assumptions are actually used, and how far we can generalise them. I had to make only minor changes there. The proofs of Lemma 3.7 and Lemma 3.9 are new, as well as the diagrams and the remark after Corollary 3.11.

3.1 A Sandwich Theorem

We start with a version of the Sandwich Theorem for topological cones. For its proof we will take advantage of existing results for ordered cones. First, we introduce sublinear and superlinear functionals:

Definition 3.1. Let C be a cone. A map $p\colon C \to \overline{\mathbb{R}}_+$ is called *sublinear* if it is homogeneous and subadditive, that is, if

$$\begin{array}{rcll} p(r \cdot a) & = & r \cdot p(a) & \qquad\text{(homogeneity)} \\ p(a+b) & \leq & p(a) + p(b) & \qquad\text{(subadditivity)} \end{array}$$

for all $a, b \in C$ and all $r \in \mathbb{R}_+$.

A map $q\colon C \to \overline{\mathbb{R}}_+$ is called *superlinear* if it is homogeneous and superadditive, that is, if

$$\begin{array}{rcll} q(r \cdot a) & = & r \cdot q(a) & \qquad\text{(homogeneity)} \\ q(a+b) & \geq & q(a) + q(b) & \qquad\text{(superadditivity)} \end{array}$$

for all $a, b \in C$ and all $r \in \mathbb{R}_+$.

We quote a sandwich theorem due to W. Roth (see [Rot00], Theorem 2.6) for ordered cones:

Theorem 3.2. *Let C be a topological cone. Let $p\colon C \to \overline{\mathbb{R}}_+$ be a sublinear and $q\colon C \to \overline{\mathbb{R}}_+$ a superlinear functional such that $a \leq b \Rightarrow q(a) \leq p(b)$. (The latter is satisfied if $q \leq p$ and one of p, q is order preserving.) Then there exists an order-preserving linear functional $\Lambda\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq \Lambda \leq p$.*

Indeed, among the order preserving sublinear functionals $f\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq f \leq p$ there are minimal ones, and each of those is linear.

Proof. We proceed in four steps.

Step 1: Without loss of generality we can assume that both p and q are order-preserving functions. Simply set $p'(a) := \inf\{p(B) \mid a \leq b\}$ and $q'(a) := \sup\{q(b) \mid b \leq a\}$. It is easy to see that p' is sublinear, q' superlinear, both are order preserving and that the estimate $q'(a) \leq p'(a)$ holds true for all $a \in C$.

Step 2: In the set of all order preserving sublinear functionals $f\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq f \leq p$ we can choose a maximal chain $\mathcal{F}$ by the Hausdorff maximality principle. The pointwise defined infimum

$$\overline{p}(x) = \inf\{f(x) \mid f \in \mathcal{F}\}$$

is again order preserving and sublinear, hence minimal in the set of all order preserving sublinear functionals $f\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq f \leq p$. In the same way, one finds an order preserving superlinear functional $\overline{q}$ which is maximal in the set of all order preserving superlinear functionals $g\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq g \leq \overline{p}$.

Step 3: Assuming that $\overline{p}$ is sublinear and order-preserving, the set

$$C' := \{a \in C \mid \overline{p}(a) < +\infty\}$$

is again a cone and a lower set in C. If μ is an order preserving linear functional below $\overline{p}$ on C', then it can be extended to a linear order preserving functional on all of C by setting it equal to $+\infty$ outside C'. For the task at hand we can therefore assume that both $\overline{p}$ and $\overline{q}$ take values below $+\infty$.

Step 4: We claim that $\overline{p} = \overline{q}$, which implies that $\Lambda := \overline{p} = \overline{q}$ is linear. For this, choose any fixed $a \in C$ and let

$$\begin{aligned} q'(x) &:= \sup\left\{ \overline{q}(c) - \overline{p}(b) \;\middle|\; b, c \in C,\ c \leq x + b \right\} \quad \text{and} \\ p'(x) &:= \inf\left\{ \overline{p}(d) + \lambda q'(a) \;\middle|\; \lambda \in \mathbb{R}_+,\ d \in C,\ x \leq d + \lambda a \right\}. \end{aligned}$$

Setting $c = x$ and $b = 0$ in the first definition we see that $\overline{q} \leq q'$, likewise by setting $d = x$ and $\lambda = 0$ in the second we have $p' \leq \overline{p}$. A simple calculation shows that superlinearity, resp. sublinearity, are preserved. By the minimality and maximality property of $\overline{p}$, resp. $\overline{q}$, we deduce $p' = \overline{p}$ and $q' = \overline{q}$. By setting $x = a$ as well as $d = 0$ and $\lambda = 1$ in the second definition we see that $p'(a) \leq q'(a)$ and this implies $\overline{p}(a) = \overline{q}(a)$ by the previous inequalities. As this is true for all $a \in C$, we conclude $\overline{p} = \overline{q}$. □

We are now heading towards a topological version of this Sandwich Theorem.

The following is well-known:

Lemma 3.3. *If $g\colon X \to \overline{\mathbb{R}}_+$ is an arbitrary function, then there is a greatest lower semicontinuous function $\check{g}\colon X \to \overline{\mathbb{R}}_+$ below g called the* lower semicontinuous envelope *of g. It is given by*

$$\check{g}(x) = \liminf g\big(\mathcal{N}(x)\big) = \sup \left\{ r \in \mathbb{R} \;\middle|\; \big(\exists U \in \mathcal{N}(x)\big)\big(\forall u \in U\big)\; r < g(u) \right\}$$

We may, in particular, form the lower semicontinuous envelope for every function $g\colon C \to \overline{\mathbb{R}}_+$, when C is a cone with a topology τ. The following properties are crucial. The lemma and its proof are due to Gordon Plotkin.

Lemma 3.4. (Plotkin) *Let C be a cone with a topology τ and let $g\colon C \to \overline{\mathbb{R}}_+$ be any function.*

(a) *If g is homogeneous and if $x \mapsto rx\colon C \to C$ is τ-continuous for every $r > 0$, then $\check{g}$ is homogeneous, too.*

(b) *If g is subadditive and if addition is continuous as a map from $(C\times C, \tau\times\tau)$ to (C, τ), then $\check{g}$ is subadditive, too.*

(c) *If g is superadditive and order preserving and if addition is almost τ-open, that is, if $\uparrow(U + V)$ is τ-open for any two τ-open sets U and V, then $\check{g}$ is superadditive, too.*

(c′) *If g is superadditive and if addition is τ-open, that is, if $U + V$ is τ-open for any two τ-open sets U and V, then $\check{g}$ is superadditive, too.*

Proof. (a) Clearly $\check{g}(sx) = s\check{g}(x)$ for $s = 0$. For $s > 0$, we note that U is an open neighbourhood of x if and only if sU is an open neighbourhood of sx. Thus, $r < \check{g}(x)$ iff $r < g(u)$ for all u in some open neighbourhood U of x iff $sr < sg(u) = g(su)$ for all u in some open neighbourhood U of x iff $sr < g(v)$ for all v in some open neighbourhood V of sx iff $sr < \check{g}(sx)$.

(b) Suppose that g is subadditive, and take $r < \check{g}(x + y)$. Then there is a τ-open W containing $x + y$ such that $g(w) > r$ for any w in W. Since addition is continuous on C, there are τ-open neighbourhoods U, V of x and y, respectively, such that $U + V$ is a subset of W. So we have for any $u \in U$ and any $v \in V$ that $r < g(u + v) \leq g(u) + g(v)$, by the subadditivity of g. Now, set $a = \inf\{g(u) \mid u \in U\}$ and $b = \inf\{g(v) \mid v \in V\}$. Then $a + b \geq r$, $\check{g}(x) \geq a$ and $\check{g}(y) \geq b$, showing that $r \leq \check{g}(x) + \check{g}(y)$.

(c) Take $r < \check{g}(x) + \check{g}(y)$. Then there are a and b such that $r < a + b$, $a < \check{g}(x)$ and $b < \check{g}(y)$. So there are τ-open neighbourhoods U, V of x and y, respectively, such that $g(u) > a$ for all $u \in U$ and $g(v) > b$ for all $v \in V$.

By hypothesis, $\uparrow(U+V)$ is τ-open, too, hence a τ-open neighbourhood of $x+y$. For all $w \in \uparrow(U+V)$ there are $u \in U$, $v \in V$ such that $w \geq u+v$, whence $g(w) \geq g(u+v) \geq g(u)+g(v) > a+b$ by the monotonicity and superadditivity of g. So we get that $\check{g}(x+y) > r$.

(c′) is proved in the same way as (c). □

Theorem 3.5. (Sandwich Theorem) *Let C be a topological cone, let $p\colon C \to \overline{\mathbb{R}}_+$ be sublinear and let $q\colon C \to \overline{\mathbb{R}}_+$ be superlinear and lower semicontinuous with $q \leq p$. Then there is a lower semicontinuous linear functional $\Lambda\colon C \to \overline{\mathbb{R}}_+$ such that $q \leq \Lambda \leq p$.*

Proof. We can apply Roth's Sandwich Theorem 3.2 to our situation with the trivial order on C. As $q \leq p$, the hypotheses of Roth's sandwich theorem are indeed satisfied. Thus, there is a linear functional Λ such that $q \leq \Lambda \leq p$. Moreover, Λ can be chosen to be minimal in the set X of all sublinear maps $s\colon C \to \overline{\mathbb{R}}_+$ with $q \leq s \leq p$. We now show that Λ is lower semicontinuous.

Lemma 3.4(a),(b) implies that $\check{\Lambda}$ is sublinear. As $q \leq \Lambda$ and as q is lower semicontinuous by hypothesis, we also have $q \leq \check{\Lambda} \leq p$. The minimality property of Λ now implies $\Lambda = \check{\Lambda}$, that is Λ is lower semicontinuous, too. □

3.2 A Separation Theorem

To prove the Separation Theorem we need the following:

Lemma 3.6. *If B is an open subset of a topological cone C then $r \cdot B$ is also open for all $r > 0$.*

Proof. This is an immediate consequence of the fact that scalar multiplication by a real number $r > 0$ is an order-isomorphism and a homeomorphism, respectively. □

Lemma 3.7. *Let M be a convex subset of a semitopological cone C. Then the closure $\overline{M}$ is convex.*

Proof. Let $M \subseteq C$ be convex. Let $m \in M$, and let $r, s > 0$ be positive scalars. We apply Proposition 1.8 to the map $x \mapsto r \cdot m + s \cdot x$, which is continuous by separate continuity of addition and scalar multiplication on C; in a second step, we use convexity of M to get

$$r \cdot m + s \cdot \overline{M} \subseteq \overline{r \cdot m + s \cdot M} \subseteq \overline{M}.$$

Since we can do this for arbitrary $m \in M$, this gives $r \cdot M + s \cdot \overline{M} \subseteq \overline{M}$ ($\bigstar$). Now we take $m \in \overline{M}$. Again, by Proposition 1.8 we obtain

$$r \cdot \overline{M} + s \cdot m \subseteq \overline{r \cdot M + s \cdot m}.$$

Since $m \in \overline{M}$, we can apply ($\bigstar$) to get

$$\overline{r \cdot M + s \cdot m} \subseteq \overline{\overline{M}} = \overline{M}.$$

Again, since $m \in \overline{M}$ was arbitrary, this gives $r \cdot \overline{M} + s \cdot \overline{M} \subseteq \overline{M}$. We have in particular: For any two elements $m, n \in \overline{M}$ and any scalar $r \in [0, 1]$, we have that $r \cdot m + (1 - r) \cdot n \in \overline{M}$. Hence $\overline{M}$ is convex. □

Definition 3.8. Let U be any subset of a topological cone C. The functional $F_U \colon C \to \overline{\mathbb{R}}_+$ defined by

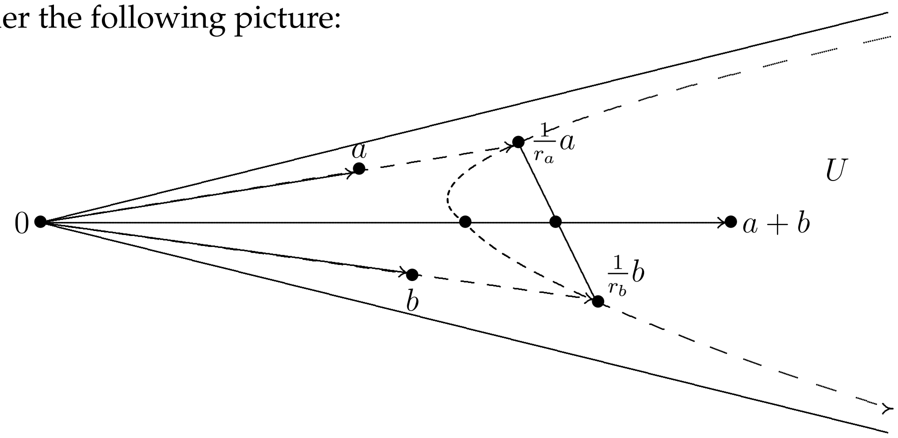
$$F_U(x) = \sup \left\{ r > 0 \;\middle|\; x \in rU \right\} = \inf \left\{ s > 0 \;\middle|\; sx \in U \right\}$$

is called the *Minkowski functional of* U.

Remark. For convenience, let us write $r_x := F_U(x)$. It is useful to observe that the Minkowski functional can be written as

$$r_x = F_U(x) = \frac{1}{\inf \left\{ r > 0 \;\middle|\; rx \in U \right\}}.$$

We will use this notation in parts (b) and (c) of the following proof of Lemma 3.9. To get a geometrical intuition for the Minkowski functional, consider the following picture:



The Minkowski functional $F_U(x)$ tells us, how much we can scale U without leaving the point x behind. The (multiplicative) inverse $\frac{1}{F_U(x)} = \frac{1}{r_x}$ tells us, how far we must stretch x at least to reach the set U.

Lemma 3.9. *Let U be an open subset of a topological cone C. Then:*

(a) *F_U is continuous and homogeneous.*

(b) *If U is convex, then F_U is superlinear.*

(c) *If $C \setminus U$ is convex, then F_U is sublinear.*

Proof.

(a) Let $]s, +\infty] \subseteq \overline{\mathbb{R}}_+$ be open. We have

$$\begin{aligned} x \in F_U^{-1}(]s, +\infty]) &\iff F_U(x) > s \\ &\iff \exists r > sx \in rU \\ &\iff x \in \bigcup_{r>s} rU \end{aligned}$$

which is open since the rU are open by Lemma 3.6. Hence the preimage is open and F_U is continuous for any open set $U \subseteq C$.

Furthermore, we obviously have

$$\begin{aligned} F_U(sx) &= \sup \left\{ r > 0 \mid sx \in rU \right\} \\ &= r \cdot \sup \left\{ r > 0 \mid x \in rU \right\} \\ &= s \cdot F_U(x), \end{aligned}$$

so F_U is also homogeneous.

(b) Suppose U is convex. Let $x, y \in C$. The points

$$\frac{1}{F_U(x)} x = \frac{1}{r_x} x \qquad \text{and} \qquad \frac{1}{F_U(y)} y = \frac{1}{r_y} y$$

lie "on the boundary" of U. That is, if we scale them by any factor greater than 1, they will lie inside U. Hence, by convexity of U, every point on the line segment connecting the two points is either inside U, or has the same property of "lying on the boundary". Since we are interested in a scaled version of $x + y$ with exactly this property, lying inside U or on the boundary makes no difference to us here. Thus, assume w.l.o.g. that the line segment between $\frac{1}{r_x} x$ and $\frac{1}{r_y} y$ is contained in U. Let μ be the amount by which we have to scale $x + y$ in order to lie on this line segment. Then we have

$$\lambda \left(\frac{1}{r_x} x \right) + (1 - \lambda) \left(\frac{1}{r_y} y \right) = \mu(x + y),$$

from which we can conclude $\frac{\lambda}{r_x} = \frac{(1-\lambda)}{r_y}$, whence $\mu = \frac{\lambda}{r_x} = \frac{1}{r_x+r_y}$. So $\frac{1}{r_x+r_y}$ is an upper bound for $\frac{1}{r_{x+y}} = \inf \{ r > 0 \mid r(x+y) \in U \}$, hence

$$\frac{1}{r_{x+y}} \leq \frac{1}{r_x + r_y}, \qquad \text{whence} \qquad r_{x+y} \geq r_x + r_y.$$

If we rewrite this abbreviation in its original form, we obtain

$$F_U(x+y) \geq F_U(x) + F_U(y),$$

which means that F_U is superlinear.

(c) If $C \backslash U$ is convex, we can apply the exact same procedure as in (b) to show that $\frac{1}{r_x+r_y}$ is a lower bound for $\frac{1}{r_{x+y}}$, hence $r_{x+y} \leq r_x + r_y$, whence $F_U(x+y) \leq F_U(x) + F_U(y)$, so F_U is sublinear.

□

Theorem 3.10. (Separation Theorem) *Let C be a topological cone with two disjoint nonempty convex subsets A and U, where U is open. Then there exists a lower semicontinuous linear functional $\Lambda \colon C \to \overline{\mathbb{R}}_+$ such that $\Lambda(a) \leq 1 < \Lambda(b)$ for all $a \in A$ and all $b \in U$.*

Proof. Without loss of generality we can assume A to be closed, since $\overline{A}$ is also nonempty convex and disjoint from U.

Let q be the Minkowski functional of U and p the Minkowski functional of $V = C \setminus A$. As $U \subseteq V$, we have $q \leq p$. By Lemma 3.9, q is superlinear, p is sublinear and both are continuous.

Now, we apply the Sandwich Theorem to get a linear lower semi-continuous function Λ with $q \leq \Lambda \leq p$. For all $a \in A$ and $b \in U$, this yields

$$\Lambda(a) \leq p(a) \leq 1 < q(b) \leq \Lambda(b),$$

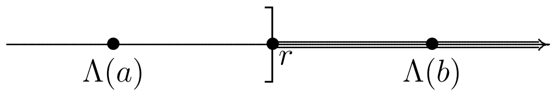
since $a \in 1 \cdot A$ implies $p(a) \leq 1$ and U open, $b = \bigsqcup^{\uparrow}_{r<1} r \cdot b$ imply that there exist a non-negative real number $r < 1$ with $r \cdot b \in U$. Thus $b \in \frac{1}{r}U$ and $\frac{1}{r} > 1$, hence, $q(b) > 1$. □

The Separation Theorem, which we just proved, implies that the lower semi-continuous linear functionals separate the points of a locally convex topological cone. More generally:

Corollary 3.11. *Let C be a locally convex topological cone and $a \not\geq b$ elements of C. Then a linear lower semi-continuous function $\Lambda \colon C \to \overline{\mathbb{R}}_+$ exists such that $\Lambda(a) < \Lambda(b)$.*

Proof. There is a convex open neighbourhood U of b such that $a \notin U$. Using this U and $A := \{a\}$, we can apply Theorem 3.10 to get the desired function Λ. □

Remark. Under the conditions of this corollary, we get an open set separating b from a in the following way:



By Corollary 3.11, to any two points $a, b \in C$ with $a \not\geq b$, there is a linear lower semicontinuous function $\Lambda\colon C \to \overline{\mathbb{R}}_+$ with $\Lambda(a) < \Lambda(b)$. Take $r \in \mathbb{R}$ with $\Lambda(a) < r < \Lambda(b)$. Then $\Lambda^{-1}(]r, +\infty])$ is convex and open by linearity and continuity of Λ, and it contains b, but not a.

3.3 A Strict Separation Theorem

We begin by considering the cone $\overline{\mathbb{R}}_+^n$ with the Scott topology. Define the *additive norm* $\|\cdot\|_1 : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ by:

$$\|x\|_1 := \sum_{i=1}^{n} x_i$$

and the *sup norm* by:

$$\|x\|_\infty := \max_{i=1,\ldots,n} x_i.$$

The additive norm is a linear continuous functional; the sup norm is sublinear and continuous, but not linear.

We say that x is *bounded* if $\|x\|_\infty < +\infty$. We have $x \gg sx$, for any bounded x and any s with $0 \leq s < 1$. (This is not true for unbounded elements.) We set $\mathbf{1} = (1, \ldots, 1) \in \overline{\mathbb{R}}_+^n$.

Lemma 3.12. *Let K be a convex Scott-compact subset of $\overline{\mathbb{R}}_+^n$ disjoint from $\downarrow\mathbf{1}$. Then there is a linear continuous functional h and an $a > 1$ such that $h(\mathbf{1}) \leq 1$ and $h(x) > a$ for all x in K.*

Proof. As $x \leq \mathbf{1}$ iff $\|x\|_\infty \leq 1$, we have $\|x\|_\infty > 1$, for any x in K. But $\|K\|_\infty$ is compact as the sup norm is continuous. So we get a b such that $+\infty > b > 1$ and $\|x\|_\infty > b$ for all x in K. Now, setting $s = \frac{1}{b}$, we get $0 < s < 1$, and, for all x in K, $sx \not\leq \mathbf{1}$. Now set

$$V = \{y \mid y \gg sx, \text{ for some } x \text{ in } K\}.$$

Clearly V is open; it is convex as K is; and it is disjoint from $\downarrow\mathbf{1}$ because $sx \not\leq \mathbf{1}$ for any x in K. So, by the Separation Theorem 3.10, there is a linear continuous functional f such that $f(x) > 1$ for x in V and $f(\mathbf{1}) \leq 1$.

The open set V contains all bounded elements of K; however it may not contain all its unbounded elements. The latter can be taken care of using the additive norm, and we combine that linearly with f to obtain h. Choose t and r such that $s < t < r < 1$, take $a = \frac{r}{t} > 1$, and set:

$$h(x) = rf(x) + (1-r)\frac{\|x\|_1}{n}.$$

Clearly $h(\mathbf{1}) \leq 1$. We claim that $h(x) > a$ holds for any x in K. For x unbounded this is immediate, since then we have $\|x\|_1 = +\infty$. For x bounded we have $tx \gg sx$ since $t > s$ and so $tx \in V$, implying $f(x) > \frac{1}{t}$. This yields that $h(x) \geq rf(x) > a$. □

Now we are ready to prove a strict separation theorem:

Theorem 3.13. (Strict Separation Theorem) *Let C be a locally convex topological cone. Suppose that K is a compact convex set and that A is a nonempty closed convex set disjoint from K. Then there is a lower semicontinuous linear functional f and an a in $\overline{\mathbb{R}}_+$ such that $f(x) > a > 1 \geq f(y)$ for all x in K and all y in A.*

Proof. Consider an element v of K. As v is not in A, by local convexity there is a convex open set U containing v and disjoint from A. So, by the Separation Theorem 3.10, there is a lower semicontinuous linear functional g such that $g(v) > 1$ and for all y in A, we have $g(y) \leq 1$. So

$$U_g := \{x \mid g(x) > 1\}$$

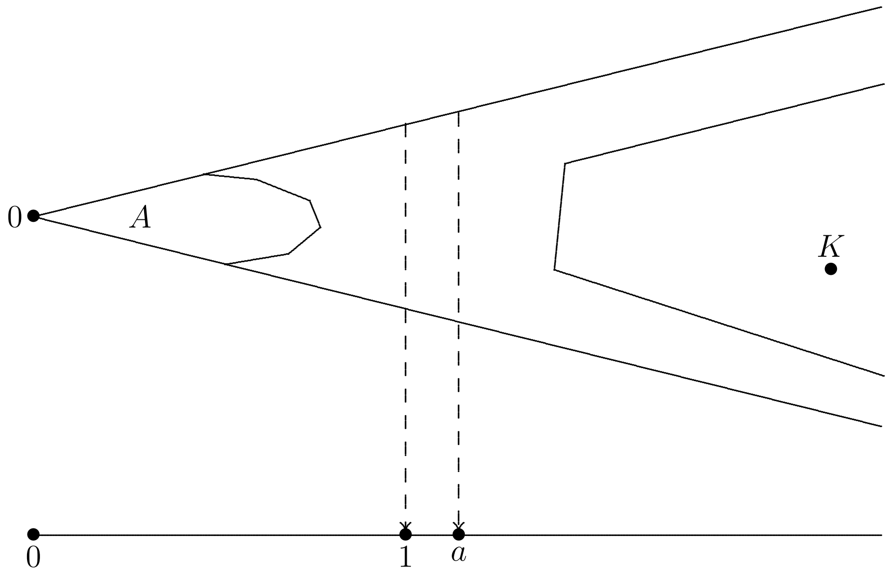
is an open set containing v and disjoint from A. As K is compact we can cover it by a finite collection $U_{g_1}, \ldots, U_{g_n}$ of such open sets. Now define $\overline{g}\colon C \to \mathbb{R}^n$ by:

$$\overline{g}(x) = (g_1(x), \ldots, g_n(x)).$$

Then $\overline{g}$ is linear and lower semicontinuous. So we have that $\overline{g}(A) \subset \downarrow\mathbf{1}$ and that $\overline{g}(K)$ is compact, convex, and disjoint from $\downarrow\mathbf{1}$ (any x in K is in some U_{g_i}, so $g_i(x) > 1$, and we have that $\overline{g}(x) \not\leq \mathbf{1}$).

Lemma 3.12 now yields a lower semicontinuous linear functional h and an $a > 1$ such that $h(\mathbf{1}) \leq 1$ and $h(x) > a$ for all $x \in \overline{g}(K)$. Choosing $f = h \circ \overline{g}$, we obtain the required functional f and constant a. □

Remark. The statement of Theorem 3.13 can be illustrated by the following picture:



Corollary 3.14. *Let C be a locally convex topological cone. Suppose that K is a compact convex set and that A is a nonempty closed convex set disjoint from K. Then they can be separated by a convex open set; that is, there is a convex open set V including K and disjoint from A.*

Proof. Take $V := \{x \in C \mid f(x) > a\}$, with f and a as given by Theorem 3.13. $\square$

4 The Convex Smyth Powercone $\mathcal{S}(C)$

The other phenomenon besides probabilistic choice that we want to model in a program is finite nondeterministic choice. As before, we can restrict ourselves to binary nondeterministic choice

$$P_1 \sqcup P_2$$

The difference to probabilistic choice is that we do not know how probable each of the possible choices is, and that this probability can change from one execution of the program to the next. The outcome of the program can be modelled by a set of possible outcomes. There are different approaches as to which subsets of the state space to allow, so there are different *powerdomain* constructions (the name indicates that we consider some subset of the power set of the state space). Some of the most prominent constructions—which have been elaborated for the case that the state space is a continuous domain—include

- the *convex lower powercone* or *convex Hoare powercone*,
- the *convex upper powercone* or *convex Smyth powercone* and
- the *biconvex powercone* or *convex Plotkin powercone*.

Whilst the convex Hoare powercone is suitable for proving partial correctness of a program, the convex Smyth powercone provides total correctness, and the convex Plotkin powercone combines both approaches. For details, see [TKP05, Chapter 4]. In this section, we will generalise as much as possible from the results of [TKP05] for the convex Smyth powercone to the case of stably compact state spaces.

4.1 Definition and Basic Properties

Definition 4.1. For a topological cone C, consider

$$\mathcal{S}(C) := \left\{ P \subseteq C \;\middle|\; P \text{ nonempty, convex, compact, saturated} \right\}$$

ordered by reverse inclusion $\supseteq$ and equipped with the following operations of addition and scalar multiplication:

$$\begin{aligned} +_{\mathcal{S}} \colon \mathcal{S}(C) \times \mathcal{S}(C) \to \mathcal{S}(C) \quad & P +_{\mathcal{S}} Q := {\uparrow}(P + Q) \\ \cdot_{\mathcal{S}} \colon \mathbb{R}_+ \times \mathcal{S}(C) \to \mathcal{S}(C) \quad & r \cdot_{\mathcal{S}} P := {\uparrow}(r \cdot P). \end{aligned}$$

On this collection consider the topology Σ generated by the following basic open sets:

$$\mathcal{N}_U := \{\, Q \in \mathcal{S}(C) \mid Q \subseteq U \,\} \qquad \text{for } U \subseteq C \text{ open.}$$

Then $\mathcal{S}(C)$ is called the *convex upper powercone* or *convex Smyth powercone* over C.

Remark. The idea behind the definition of Σ is to carry over the topology on the starting space to the powercone: We take the original open sets in C and collect all elements of $\mathcal{S}(C)$ that lie inside them.

Moreover, the sets $\mathcal{N}_U$ do indeed form a basis of a topology on $\mathcal{S}(C)$, since clearly $\mathcal{N}_U \cap \mathcal{N}_V = \mathcal{N}_{U\cap V}$ and $U \cap V$ is open if U and V are.

Theorem 4.2. *The collection $\mathcal{S}(C)$ with the operations $+_{\mathcal{S}}$ and $\cdot_{\mathcal{S}}$ and the topology Σ is a topological cone.*

We will prove this theorem in a number of steps.

Proposition 4.3. *Scalar multiplication on $\mathcal{S}(C)$ is jointly continuous with respect to the Scott topology on $\mathbb{R}_+$ and Σ on $\mathcal{S}(C)$.*

Proof. We omit the verification that scalar multiplication is indeed well-defined, as the procedure of [TKP05, Proof of Proposition 4.13] where C is a dcpo-cone, can be directly taken over for the case that C is a stably compact and locally convex topological cone. We show that $\cdot_{\mathcal{S}}$ is continuous in each of its two components. By the remark following Definition 1.23, this suffices to prove joint continuity.

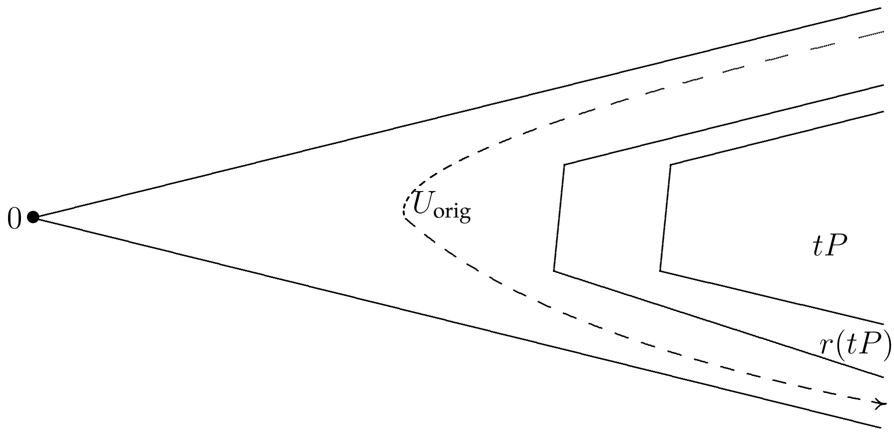
- $s_r\colon \mathcal{S}(C) \to \mathcal{S}(C), P \mapsto \uparrow(rP)$ is continuous: Suppose $U \subseteq \mathcal{S}(C)$ is open w.r.t. Σ. If $r = 0$, then $\uparrow(0\cdot P) = C$ for all $P \in \mathcal{S}(C)$. But then, we have $s_0^{-1}(U) = \mathcal{S}(C)$ if $C \in U$ and $s_0^{-1}(U) = \varnothing$ if $C \notin U$, both of which are trivially open. Thus, assume $r > 0$. Then $s_r^{-1} = \left\{\frac{1}{r}P \mid P \in U\right\}$, which is open—for let $U = \bigcup_i \mathcal{N}_{U_i}$ be formed by some basic open sets via arbitrary union. Then $s_r^{-1}(U) = \bigcup_i \mathcal{N}_{\frac{1}{r}U_i}$.

- $s_P\colon \mathbb{R}_+ \to \mathcal{S}(C), r \mapsto \uparrow(rP)$ is continuous: Suppose $U \subseteq \mathcal{S}(C)$ is open w.r.t. Σ, and consider $s_P^{-1}(U) = \{r \in \mathbb{R}_+ \mid \uparrow(rP) \in U\}$. There are two cases: If there is no $r \in \mathbb{R}_+$ with $\uparrow(rP) \in U$, then $s_P^{-1}(U)$ is empty, hence open. Hence, assume there is $r_0 \in \mathbb{R}_+$ with $\uparrow(r_0P) \in U$. If $r_0 = 0$, then $U = \mathcal{S}(C)$ and $s_P^{-1}(U) = \mathbb{R}_+$, which is trivially open. Suppose henceforth that $r_0 > 0$ for all $r_0 \in s_P^{-1}(U)$, hence we have $r_0P = \uparrow(r_0P)$. In $\mathbb{R}_+$, the open sets are of the form $]s, +\infty[$. To show that s_P^{-1} is such an interval, it suffices to show that it is an upper set and that for any $t \in s_P^{-1}(U)$, there is $t' \in s_P^{-1}(U)$ with $t' < t$. We will call the latter property ($\star$). First we show that the preimage is an upper set. Consider an element $r \in s_P^{-1}(U)$ and take some element $r' > r$ above it. From $rP \in U$ and

$$r'P = \underbrace{\left(\frac{r'}{r}\right)}_{>1} \cdot rP \subseteq rP,$$

we conclude that $r'P$ lies above rP in the order of reverse inclusion. Since U is an open, hence upper set, this means that $r'P$ also belongs to U.

It remains to show that the preimage of U fulfils property ($\star$). So let $t \in s_P^{-1}(U)$, i.e. $tP \in U$. For any $t' < t$, we have $t'P \supseteq tP$. We now seek a particular t' below t such that $t'P \in U$. Equivalently, we seek $0 < r < 1$ such that $(rt)P = r(tP) \in U$ and then set $t' := rt$. That is, we want to scale down the compact set tP by a small amount r, such that the scaled version is still contained in U. Now, we recall how the open sets in Σ are constructed from the basic sets. Since tP is an element in the open set U, there must be a neighbourhood $U_{\text{orig}} \subseteq C$ of $tP \subseteq C$ which is open in the original topology of C. So we want to scale down tP by r such that the scaled set is still contained in U_{orig}, as illustrated in the following picture:



We have that $r(tP)$ is an element in U, if we can show that there is $0 < r < 1$ such that $r(tP)$ is still contained in U_{orig}. To this end, consider

$$f\colon C \to \overline{\mathbb{R}}_+ \qquad \text{with} \qquad x \mapsto f(x) := \inf \left\{ s \in \overline{\mathbb{R}}_+ \;\middle|\; sx \in U_{\text{orig}} \right\},$$

where $\overline{\mathbb{R}}_+$ is equipped with the upper topology, whose non-trivial open sets are of the form $[0, k[$, for $k \in \overline{\mathbb{R}}_+$. This function is continuous, since $f^{-1}([0, k[) = \frac{1}{k} U_{\text{orig}}$ is open in C by Lemma 3.6, and $f^{-1}(\varnothing) = \varnothing$ as well as $f^{-1}(\overline{\mathbb{R}}_+) = C$ are open, too. Since C is a topological cone, scalar multiplication $\cdot_x$ by a fixed element $x \in tP \subseteq C$ is continuous, hence $\left\{s \in \mathbb{R}_+ \mid sx \in U_{\text{orig}}\right\} = \cdot_x^{-1}(U_{\text{orig}})$ is open in $\mathbb{R}_+$ with the Scott topology. For $x \in tP \subseteq U_{\text{orig}}$, we know $1 \in \cdot_x^{-1}(U_{\text{orig}})$, so $\cdot_x^{-1}(U_{\text{orig}}) = [k, +\infty[$ with $k < 1$. Using this, we are almost done: We infer $f(x) = \inf \cdot_x^{-1}(U_{\text{orig}}) < 1$ for $x \in tP$, hence $f(tP) \subseteq [0, 1[$.

For the last step, observe that continuous functions map compact path-connected sets to compact path-connected sets. Clearly, tP is path-connected, since it is convex, and it is compact by assumption. So $f(tP)$ is path-connected and compact in $\overline{\mathbb{R}}_+$ with the upper topology. In $\overline{\mathbb{R}}_+$, the path-connected sets are exactly the intervals. Furthermore, $f(tP)$ has to be a bounded interval, since we have seen above that it is strictly bounded by 1. It is easy to show that intervals of the form $[k_1, k_2[$ or $]k_1, k_2[$ are not compact in the upper topology. Hence, $f(tP)$ must be of the form $[k_1, k_2]$ or $]k_1, k_2]$. In both cases, we see that f takes on its maximum on the set tP, and that this maximum is strictly bounded by 1. Set $n := \max f(tP) < 1$, then $r := \frac{n+1}{2}$ is the sought number. It is strictly smaller than 1, and it can be used to scale any element of tP without leaving U_{orig}. This completes the proof that scalar multiplication with a fixed element P in $\mathcal{S}(C)$ is continuous.

□

Proposition 4.4. *Addition on $\mathcal{S}(C)$ is jointly continuous with respect to Σ.*

Proof. As above, we omit the verification that $+_{\mathcal{S}} \colon \mathcal{S}(C) \times \mathcal{S}(C) \to \mathcal{S}(C)$ is well-defined and refer to [TKP05, Proof of Proposition 4.13].

Now, to show continuity, let $P, Q \in \mathcal{S}(C)$. Let $\mathcal{N}_U \subseteq \mathcal{S}(C)$ be basic open with $P +_{\mathcal{S}} Q \in \mathcal{N}_U$. We have

$$\begin{aligned}
P +_{\mathcal{S}} Q \in \mathcal{N}_U &\iff P +_{\mathcal{S}} Q \subseteq U \\
&\iff {\uparrow}(P + Q) \subseteq U \\
&\iff P + Q \subseteq U && \text{since } U \text{ is open hence upper} \\
&\iff P \times Q \subseteq +^{-1}(U)
\end{aligned}$$

We have that P and Q are compact and that $+^{-1}(U)$ is open by continuity of $+$ on C. So we can apply an argument from general topology (see

[Kel55, Chapter 5, Theorem 12]) to get open sets U_1 and U_2 with $P \subseteq U_1$ and $Q \subseteq U_2$ and $U_1 \times U_2 \subseteq +^{-1}(U)$. So we get open neighbourhoods $P \in \mathcal{N}_{U_1}$ and $Q \in \mathcal{N}_{U_2}$ with

$$\mathcal{N}_{U_1} + \mathcal{N}_{U_2} \subseteq \uparrow(\mathcal{N}_{U_1} + \mathcal{N}_{U_2}) = \mathcal{N}_{U_1} +_{\mathcal{S}} \mathcal{N}_{U_2} \subseteq \mathcal{N}_U.$$

Hence, addition $+_{\mathcal{S}}$ on $\mathcal{S}(C)$ is jointly continuous. $\square$

Finally, we have to verify the cone axioms. We state a simple lemma first.

Lemma 4.5. *In a cone C, the equality $(r+s)P = rP+sP$ holds for any convex subset $P \subseteq C$ and any scalars $r, s \geq 0$.*

Proof. For the inclusion "$\subseteq$", let $(r+s) \cdot x \in (r+s) \cdot P$. We have

$$(r+s) \cdot x = rx + sx \in rP + sP$$

For the second inclusion "$\supseteq$", let $rx + sx' \in r \cdot P + s \cdot P$. We have

$$\begin{aligned} rx + sx' &= \tfrac{r+s}{r+s} \cdot (rx + sx') \\ &= (r+s) \cdot \left(\tfrac{1}{r+s} \cdot rx + \tfrac{1}{r+s} \cdot sx'\right) \\ &= (r+s) \cdot \left(\tfrac{r}{r+s} \cdot x + \tfrac{s}{r+s} \cdot x'\right) \\ &\in (r+s) \cdot P \qquad\qquad \text{as } \tfrac{r}{r+s} + \tfrac{s}{r+s} = 1 \text{ and } P \text{ is convex} \end{aligned}$$

which finishes the proof. $\square$

Proposition 4.6. *The triple $\big(\mathcal{S}(C), +_{\mathcal{S}}, \cdot_{\mathcal{S}}, C\big)$ is a cone.*

Proof. We only outline most of the verifications, since they are straightforward. Addition on $\mathcal{S}(C)$ is associative since addition on C is monotone and associative. It is commutative since addition on C is commutative. The entire cone C as a set is the neutral element for addition on $\mathcal{S}(C)$, since $0 \in C$ and since addition $+$ is monotone. We have $1 \cdot_{\mathcal{S}} P = P$ and $0 \cdot_{\mathcal{S}} P = C$, which is the neutral element for addition. We have $(r \cdot s) \cdot_{\mathcal{S}} P = r \cdot_{\mathcal{S}} (s \cdot_{\mathcal{S}} P)$ using monotonicity of scalar multiplication on C. The equation $r \cdot_{\mathcal{S}} (P +_{\mathcal{S}} Q) = (r \cdot_{\mathcal{S}} P) +_{\mathcal{S}} (s \cdot_{\mathcal{S}} Q)$ follows from monotonicity of addition and scalar multiplication as well as the corresponding distributivity law on C. With the additional help of Lemma 4.5, the proof of the second distributivity law, $(r+s) \cdot_{\mathcal{S}} P = (r \cdot_{\mathcal{S}} P) +_{\mathcal{S}} (s \cdot_{\mathcal{S}} P)$, is straightforward. $\square$

We have shown that the convex Smyth powercone over a topological cone is again a topological cone. Throughout the remainder of this subsection, unless otherwise stated, we will consider the convex Smyth powercone $\mathcal{S}(C)$ over a stably compact locally convex topological cone C.

Proposition 4.7. *Let C be a stably compact topological cone, and $\mathcal{S}(C)$ the convex Smyth powercone over C. Then the topology Σ satisfies the T_0 separation axiom.*

Proof. Take $P, Q \in \mathcal{S}(C)$ with $P \not\leq Q$, i.e. $Q \not\subseteq P$. We are looking for an open set containing P but not Q. By assumption, $Q \not\subseteq P$, hence we can find $q \in Q \setminus P$. In C, set

$$U := C \setminus \overline{\{ q \}} = C \setminus \downarrow q.$$

This is an open subset with $P \subseteq U$ but $Q \not\subseteq U$ by construction. Thus, we have $P \in \mathcal{N}_U$ but $Q \notin \mathcal{N}_U$. □

Now we will give a characterisation of the way below relation on the convex Smyth powercone. The corresponding version in the case of the *classical Smyth powerdomain* defined by

$$\mathcal{S}_c(X) := \left\{ P \subseteq X \;\middle|\; P \text{ nonempty, compact, saturated} \right\}$$

for any topological space X (so in particular for any topological cone) can be found in [GHK+03, Prop. I-1.24.2(ii), p.67]. The new proof for nonempty *convex* compact saturated sets is inspired by the one for the classical case, but makes use of the additional cone structure and the separation theorems in Section 3.

Remark. Note that the classical Smyth powerdomain $\mathcal{S}_c(X)$ over a topological space X is denoted by $Q^*(X)$ in [GHK+03].

Proposition 4.8. *Let C be any topological cone, $K_1, K_2 \in \mathcal{S}(C)$ and consider the following assertions:*

(a) *there is a convex open set $U \subseteq C$ such that $K_2 \subseteq U \subseteq K_1$*

(b) *$K_1 \ll K_2$ in $\mathcal{S}(C)$*

Then we have (a) $\Longrightarrow$ (b) if C is well-filtered; if, in addition, C is coherent, locally compact and locally convex, then (b) $\Longrightarrow$ (a) holds and the two properties are equivalent.

Before we can prove this, we need a lemma from [TKP05]. We cite it together with its proof, because it is short and elegant and enlightening:

Lemma 4.9. ([TKP05, Lemma 2.9]) *For compact convex subsets P and Q of a topological cone,* $\operatorname{conv}(P \cup Q)$ *is also compact.*

Proof. The set $\Delta_2 = \left\{ (r, 1-r) \,\middle|\, r \in [0,1] \right\}$ is compact with respect to the Scott topology on $[0,1]^2$. The map from $\Delta_2 \times C \times C$ to C, defined by $\big((r,s), x, y\big) \mapsto r \cdot x + s \cdot y$ is continuous. The convex hull of $P \cup Q$ is equal to the image of the compact set $\Delta_2 \times P \times Q$. Thus, $\operatorname{conv}(P \cup Q)$ is also compact. □

Remark. Note that the convex hull of a compact set is *not* compact in general. Hence, we can not drop the condition that the compact sets P and Q be convex in the previous lemma.

Now we are ready to prove Proposition 4.8:

Proof of Proposition 4.8. First, we show (a) $\Longrightarrow$ (b), assuming that C is well-filtered:

Let $\mathcal{C}$ be a filter base of convex compact saturated subsets such that $\bigcap \mathcal{C} \subseteq K_2$, i.e. $K_2 \sqsubseteq \bigcap \mathcal{C}$ in the order of reverse inclusion. Since by assumption (a) we have a convex open set U with $K_2 \subseteq U$, the well-filteredness property provides us with a $C \in \mathcal{C}$ such that $C \subseteq K_2$, hence $C \subseteq K_1$, which means $K_1 \sqsubseteq C$ in the given order. So we have shown $K_1 \ll K_2$.

Now, suppose in addition that C is locally convex and locally compact to show the implication (b) $\Longrightarrow$ (a):

We begin by proving that K_2 is the intersection of convex compact neighbourhoods. This means that for any $y \notin K_2$ we must find some convex compact neighbourhood U_y of K_2 that does not contain y. Consider $\downarrow y = \overline{\{y\}}$, which is closed and convex by Lemma 3.7. By Corollary 3.11 resp. the remark following it, we get an open convex set C_y that separates K_2 from $\downarrow y$. Since C_y is a neighbourhood for each point $x \in K_2$, by local compactness we find a compact neighbourhood C_x for every $x \in K_2$ such that C_x is contained in C_y. These C_x cover K_2; in particular, their interiors form an open cover. Since K_2 is compact, we find $x_1, \ldots, x_n$ such that

$$K_2 \subseteq \underbrace{C_{x_1} \cup \cdots \cup C_{x_n}}_{=:N}.$$

Since the union is finite, N is compact, its convex hull $\operatorname{conv}(N)$ is then compact by Lemma 4.9. Since C_y was convex and the C_x are all contained in it, we have $\operatorname{conv}(N) \subseteq N \subseteq C_y$. Hence, $\operatorname{conv}(N)$ is the U_y we were looking for.

Now, K_2 is the intersection of a collection of convex compact neighbourhoods. We know that the intersection of two convex sets is convex, by coherence we have that the intersection of two compact saturated sets is compact, so we can extend the given collection to a filter base $\mathcal{C}$ with

$$K_2 = \bigcap \mathcal{C} = \bigsqcup^{\uparrow} \mathcal{C}.$$

Since by assumption we have $K_1 \ll K_2$, we find P in $\mathcal{C}$ with $K_1 \sqsubseteq P$, i.e. $P \subseteq K_1$. Since P is a neighbourhood of K_2, we have

$$K_2 \subseteq \mathrm{int}(P) \subseteq P \subseteq K_1.$$

However, we are not yet finished, for P is open but need not be convex.

We have $\bigcap \mathcal{C} = K_2 \subseteq \mathrm{int}(P)$, hence by well-filteredness, there is $Q_1 \in \mathcal{C}$ with $Q_1 \subseteq \mathrm{int}(P)$. Since Q_1 is again compact and convex, we can repeat the argument made above for K_2 to find that $Q_1 = \bigcap \mathcal{C}_1$ holds for some filter base $\mathcal{C}_1$ of compact convex neighbourhoods of Q_1. Again, we use well-filteredness and $\bigcap \mathcal{C}_1 = Q_1 \subseteq \mathrm{int}(P)$ to get a neighbourhood $Q_2 \in \mathcal{C}_1$ of Q_1 with $Q_2 \subseteq \mathrm{int}(P)$. Repeating this procedure gives an increasing sequence of compact convex neighbourhoods $Q_i \subseteq \mathrm{int}(Q_{i+1})$:

$$K_2 \subseteq Q_1 \subseteq Q_2 \subseteq \ldots \subseteq \mathrm{int}(P).$$

We form the union of these neighbourhoods:

$$U := \bigcup_{i \geq 1} Q_i.$$

Since each Q_i is contained in $\mathrm{int}(P)$, so is the union. This is an increasing, hence directed union, and its members are convex. Hence the union is convex. Finally, the set U is a neighbourhood for any of its elements. To see this, let $x \in U$. It is contained in some Q_i. Since we have

$$x \in Q_i \subseteq \mathrm{int}(Q_{i+1}) \subseteq U,$$

we have that U is a neighbourhood of x. This shows that U is open, hence we have found a convex open set with $K_2 \subseteq U \subseteq K_1$. □

From results on the classical Smyth powerdomain $\mathcal{S}_c(C)$ in [GHK+03, Prop. I-1.24.2(iii), p.67], it follows immediately that the convex Smyth powercone $\mathcal{S}(C)$ is a dcpo whenever C is a well-filtered topological space (since $\mathcal{S}_c(C)$ is a dcpo in this case and convex sets are closed under arbitrary intersections). To show that it is even a continuous domain is slightly more complicated.

Theorem 4.10. *Let C be a stably compact locally convex topological cone. Then the convex Smyth powercone $\mathcal{S}(C)$ is a bounded complete domain, so in particular a continuous domain.*

The proof will go along these lines: We will use certain results from [GHK+03] about the classical Smyth powerdomain. To go from one construction to the other and back again, we will use the convex hull operator $\mathrm{conv}\colon \mathcal{S}_c(C) \to \mathcal{S}(C)$ and the canonical injection $i\colon \mathcal{S}(C) \to \mathcal{S}_c(C)$.

$$
\begin{tikzcd}
\mathcal{S}_c(C) = \{\, A \subseteq C \mid A \text{ nonempty compact saturated} \,\} \arrow[d, shift right, "\mathrm{conv}"'] \\
\mathcal{S}(C) \ = \{\, A \subseteq C \mid A \text{ nonempty compact saturated convex} \,\} \arrow[u, hook, shift right, "i"']
\end{tikzcd}
$$

We will show that the composition of both is a Scott-continuous *projection*, i.e. a monotone, idempotent selfmap, and since $\mathcal{S}(C)$ is its image, we can apply [GHK+03, Theorem I-2.2, p.80] to conclude that the convex Smyth powercone is a continuous domain.

Lemma 4.11. *Consider the convex hull operator defined by*

$$\mathrm{conv}\colon \mathcal{S}_c(C) \to \mathcal{S}(C) \qquad \textit{with} \qquad Q \mapsto \mathrm{conv}(Q),$$

consider the canonical injection defined by

$$i\colon \mathcal{S}(C) \to \mathcal{S}_c(C) \qquad \textit{with} \qquad Q \mapsto Q.$$

Consider the composition $i \circ \mathrm{conv}\colon \mathcal{S}_c(C) \to \mathcal{S}_c(C)$ with $Q \mapsto \mathrm{conv}(Q)$. Then we have:

(a) *The selfmap $i \circ \mathrm{conv}\colon \mathcal{S}_c(C) \to \mathcal{S}_c(C)$ is a projection operator*

(b) *The selfmap $i \circ \mathrm{conv}\colon \mathcal{S}_c(C) \to \mathcal{S}_c(C)$ is Scott-continuous, i.e. it preserves directed suprema:*

$$\bigsqcup^{\uparrow} \mathrm{conv}(\mathcal{C}) = \bigcap \mathrm{conv}(\mathcal{C}) = \mathrm{conv}\left(\bigcap \mathcal{C}\right) = \mathrm{conv}\left(\bigsqcup^{\uparrow} \mathcal{C}\right)$$

for every directed set $\mathcal{C}$.

Proof. Clearly, we have $\mathrm{conv}(P) = \mathrm{conv}(\mathrm{conv}(P))$, and if $P \subseteq Q$ then $\mathrm{conv}(P) \subseteq \mathrm{conv}(Q)$. So $i \circ \mathrm{conv}$ is idempotent and monotone. For Scott continuity, let $\mathcal{C}$ be a directed set in $\mathcal{S}_c(C)$, i.e. a filter base of nonempty compact saturated sets in C. We want to show:

$$\operatorname{conv}\left(\bigcap\{\, K \subseteq C \mid K \in \mathcal{C}\,\}\right) = \bigcap\{\, \operatorname{conv}(K) \mid K \in \mathcal{C}\,\}.$$

"$\supseteq$": Clear, since $\operatorname{conv}(K) = \bigcap\{P \subseteq C \mid K \subseteq P \text{ convex}\}$ and $\bigcap \mathcal{C} \subseteq K$ for every $K \in \mathcal{C}$.

"$\subseteq$": Let $x \in \operatorname{conv}(\bigcap \mathcal{C})$. Then $x = \lambda x_1 + (1-\lambda)x_2$ for some $x_1, x_2 \in \bigcap \mathcal{C}$. But then $x_1, x_2 \in K$ for every K in $\mathcal{C}$, so $\lambda x_1 + (1-\lambda)x_2$ is in $\operatorname{conv}(K)$ for every K in $\mathcal{C}$, so $x \in \bigcap\{\operatorname{conv}(K) \mid K \in \mathcal{C}\}$. □

With this technical lemma, we are ready to prove Theorem 4.10:

Proof of Theorem 4.10. By [GHK+03, Prop. I-1.24.2(iv), p.67], we have that $\mathcal{S}_c(C)$ is a continuous domain, and by Lemma 4.11, we have that $\mathcal{S}(C)$ is its image under the Scott-continuous projection operator $i \circ \operatorname{conv}$. Now, by [GHK+03, Theorem I-2.2, p.80], such images are continuous domains, too. □

Proposition 4.12. *Let C be a stably compact locally convex topological cone. Then the topology Σ is equal to the Scott topology on $\mathcal{S}(C)$.*

Proof. First we show that the Scott topology is finer than Σ:

Let $U \in \Sigma$, let $\mathcal{A} \subseteq \mathcal{S}(C)$ be directed with $\bigcap \mathcal{A} = \bigsqcup^{\uparrow} \mathcal{A} \in U$. Since the empty set does not belong to $\mathcal{S}(C)$, we have that $\mathcal{A}$ is a filter base on C. Now, $\bigsqcup^{\uparrow} \mathcal{A}$ is an element in the open set U, so it is an element in some basic open set $\mathcal{N}_V$ contained in U. By the definition of the basic open sets, this means that V is open and $\bigcap \mathcal{A} \subseteq V$ in C. Since $\bigcap \mathcal{A}$ is an element of the convex Smyth powercone, it is nonempty. So we can apply the well-filteredness property to find an $A \in \mathcal{A}$ with $A \subseteq V$. Thus, we obtain $A \in \mathcal{N}_V \subseteq U$, so the set U is Scott-open.

Next, we show that Σ is finer than the Scott topology:

Let U be a basic Scott-open set in $\mathcal{S}(C)$. If $U = \varnothing$, then we are done. Hence, assume $U \neq \varnothing$. Since $\mathcal{S}(C)$ is a continuous domain, the sets $\twoheaduparrow Q$ for elements Q are a basis of the Scott topology. Hence, we have $U = \twoheaduparrow Q$ with $Q \in \mathcal{S}(C)$. The space C is stably compact and locally convex, so by Proposition 4.8, we have that $Q_i \gg Q$ is equivalent to $Q_i \subseteq U_i \subseteq Q$ for some convex open set U_i. Hence, we have that $U = \bigcup_i \mathcal{N}_{U_i}$ is an open set w.r.t. the topology Σ. □

Combining this result with our main Theorem 4.2, we conclude:

Corollary 4.13. *Let C be a stably compact locally convex topological cone. Then the convex Smyth powercone $\mathcal{S}(C)$, equipped with the Scott topology, is a continuous dcpo-cone.*

We finish this subsection by showing that the properties of being stably compact and locally convex are inherited from C to $\mathcal{S}(C)$.

Proposition 4.14. *Let C be a stably compact locally convex topological cone. Then the convex Smyth powercone $\mathcal{S}(C)$ with the Scott topology is stably compact and locally convex.*

Proof. Since by Corollary 4.13, $\mathcal{S}(C)$ is a continuous dcpo-cone, it is automatically locally convex with the Scott topology by [TKP05, Prop. 2.5].

The proof that $\mathcal{S}(C)$ is stably compact combines several results from [GHK$^+$03]. Unfortunately, it uses several concepts that we have not introduced here, because we do not use them in this paper. For those who are familiar with domain and lattice theory, I will state the proof using all necessary vocabulary. The readers who are not so well acquainted with the subject may consider this proposition as an external result of its own.

From [GHK$^+$03, Prop. I-1.24.2(iv), p.67], we know that $\mathcal{S}_c(C)$ is a continuous semilattice. So in particular, it is a continuous domain, and certainly every bounded finite set has a supremum in $\mathcal{S}_c(C)$: By coherence of C, every finite intersection of compact sets is compact; if the sets are bounded above, this finite intersection cannot be empty, so it is an element of $\mathcal{S}_c(C)$.

Now, by [GHK$^+$03, Prop. I-1.25, p.69], this implies that the poset $\mathcal{S}_c(C)$ is a bounded complete continuous domain (note that in [GHK$^+$03], a continuous domain is simply called 'domain'). Using Lemma 4.11 as well as [GHK$^+$03, Theorem I-2.2, p.80], we see that $\mathcal{S}(C)$ is the image of the bounded complete domain $\mathcal{S}_c(C)$ under the Scott-continuous projection operator $i \circ \mathrm{conv}$, hence it is a bounded complete domain, too.

But bounded complete domains are complete continuous semilattices, so we can use [GHK$^+$03, Theorem III-1.9, p.214] to conclude that the Lawson topology on $\mathcal{S}(C)$ is compact. By [GHK$^+$03, Prop. VI-6.24, p.482], a continuous Lawson-compact semilattice is stably compact both with the lower and with the Scott topology. $\square$

4.2 A property for the composition of programs

Interpreting a program P as a function $[\![P]\!] \colon C \to \mathcal{S}(C)$, we can express the fact that nondeterministic choice has taken place—since the output is not a single value but a set of possible outcomes, with no priority on them. What we still need is a way to model every single nondeterministic choice during the program. Naïvely, if for the given state space, programs P_1 and P_2 can have the output Q, resp. R, and we make a nondeterministic choice $P_1 \sqcup P_2$, then the possible outcomes are now $Q \cup R$. Now, if we also remember that we have to stay inside the convex Smyth powerdomain, which consisted of compact saturated convex subsets of the state space, this gives us the following notion of infima:

Proposition 4.15. *Let C be any topological cone. Then binary infima exist in $\mathcal{S}(C)$ and are given by*

$$P \wedge Q = \uparrow \mathrm{conv}(P \cup Q).$$

Furthermore, the infima have the following properties:

$$\begin{aligned} P +_{\mathcal{S}} (Q \wedge R) &= (P +_{\mathcal{S}} Q) \wedge (P +_{\mathcal{S}} R) \\ r \cdot_{\mathcal{S}} (P \wedge Q) &= (r \cdot_{\mathcal{S}} P) \wedge (r \cdot_{\mathcal{S}} Q). \end{aligned}$$

Proof. The union of two compact sets is always compact, and by Lemma 4.9, its convex closure is still compact. Taking the saturation of this closure changes neither the property of being compact nor the property of being convex.

The properties of distributing over $+_{\mathcal{S}}$ and $\cdot_{\mathcal{S}}$ are shown to hold in [TKP05, Proof of Lemma 4.15] for the case of dcpo-cones. Since no particular features distinguishing dcpo-cones from our topological cones are used there, the proof goes through in our context as well.. The Lemma 2.8 which is used therein is stated in the same paper for arbitrary cones. □

Just like for the case of the extended probabilistic powerdomain, an extension property is needed to ensure that the composition of programs can be given a proper semantic. The first step in doing this is to extend the assignment $C \mapsto \mathcal{S}(C)$ to a functor.

Proposition 4.16. *Let $f\colon C \to D$ be a continuous linear map between stably compact locally convex topological cones. Then the map $\mathcal{S}(f)\colon \mathcal{S}(C) \to \mathcal{S}(D)$, defined by*

$$\mathcal{S}(f)(P) := \uparrow f(P)$$

is a linear continuous function preserving binary infima.

Proof. The corresponding proof for the case of continuous dcpo-cones in [TKP05, Prop. 4.19] can be overtaken as is: By Corollary 4.13, we have that $\mathcal{S}(C)$ and $\mathcal{S}(D)$ are continuous dcpo-cones themselves, and the fact that in continuous dcpo-cones, every compact convex saturated set is the intersection of its compact convex saturated neighbourhoods has been shown to hold true for locally convex locally compact topological cones in the proof of Proposition 4.8. □

With this, we have shown:

Corollary 4.17. *The convex Smyth powercone operator $\mathcal{S}$ is a functor from the category* $\mathbf{StCpCvCone}$ *of stably compact locally convex topological cones into the category* $\mathbf{StCpCvCone}^{\wedge}$ *of stably compact locally convex topological cones with binary infima.*

Now we have to define natural transformations η and μ such that the triple $(\mathcal{S}, \eta, \mu)$ becomes a monad in the category $\mathbf{StCpCvCone}$ of stably compact locally convex topological cones. For a given cone C in this category, define $\eta_C \colon C \to \mathcal{S}(C)$ by

$$\eta_C(x) := \uparrow x.$$

It is straightforward to check that η_C is well-defined, i.e. that $\uparrow x$ is non-empty, compact and convex. For a convex compact saturated subset Q of $\mathcal{S}\mathcal{S}(C)$, define $\mu_C \colon \mathcal{S}\mathcal{S}(C) \to \mathcal{S}(C)$ by

$$\mu_C(\mathcal{K}) = \bigcup \{ P \subseteq C \mid P \in \mathcal{K} \}.$$

for $\mathcal{K} \in \mathcal{S}\mathcal{S}(C)$. Then we have:

Proposition 4.18. *The maps μ_C and η_C are well-defined and $(\mathcal{S}, \eta, \mu)$ is a monad. For a given morphism $f \colon C \to D$ in* $\mathbf{StCpCvCone}$*, the function*

$$\hat{f} := \mu_D \circ \mathcal{S}(f)$$

is a linear lower semicontinuous function preserving binary infima, such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & \mathcal{S}(C) \arrow[d, dashed, "\hat{f}"] \arrow[dr, "\mathcal{S}(f)"] & \\
& \mathcal{S}(D) & \mathcal{S}\mathcal{S}(D) \arrow[l, "\mu_D"]
\end{tikzcd}$$

Proof. As is clear from the monad properties, we have $\mu \circ \mathcal{S}\eta = \mathsf{Id}_{\mathbf{StCpCone}}$, hence $\mu_D \circ \mathcal{S}(f) \circ \eta_C = f$. The verifications that have to be made for the proof of this proposition are lengthy but straightforward and shall not be carried out in explicit detail here. □

The function $\hat{f}$ is the sought-after extension to f that we need for the composition of programs. Note that, here again, we have shown a property which is more general than we need, since in the concrete case of program composition, we have $C = D$. As in Section 2, the extension $\hat{f}$ has a certain uniqueness property which is—again—irrelevant for our applications.

We finish this section with the following special case of our extension property:

Proposition 4.19. *For every stably compact topological cone C, we have the following property:*

For every lower semicontinuous function $f\colon C \to \overline{\mathbb{R}}_+$, there is a lower semicontinuous linear function $\hat{f}\colon \mathcal{S}(C) \to \overline{\mathbb{R}}_+$ preserving binary infima, such that $\hat{f} \circ i_C = f$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook, "i_C"] \arrow[dr, "f"'] & \mathcal{S}(C) \arrow[d, "\hat{f}"] \\ & \overline{\mathbb{R}}_+ \end{tikzcd}$$

This function is given by $\hat{f}(Q) := \inf f(Q)$.

Proof. Clearly, $\hat{f}$ is well-defined: Since f is continuous and monotone, compact saturated sets Q are mapped to compact saturated sets $f(Q)$ in $\overline{\mathbb{R}}_+$. These are of the form $[r, +\infty]$, hence $r = \inf f(Q)$.

Furthermore, we have

$$\hat{f}(i_C(x)) = \hat{f}(\uparrow x) = \inf \underbrace{f(\uparrow x)}_{\ni f(x)} = f(x),$$

thus the diagram commutes.

We still have to check that $\hat{f}$ is continuous and linear and that it preserves binary infima. First, notice that $\hat{f}$ is continuous if and only if it is Scott-continuous, since we have the Scott topology on $\mathcal{S}(C)$ and $\overline{\mathbb{R}}_+$. So we have to show that $\hat{f}$ preserves directed suprema. We have

$$\begin{aligned}\hat{f}\Big(\bigvee_i^{\uparrow} P_i\Big) = \inf f\Big(\bigvee_i^{\uparrow} P_i\Big) &= \inf f\Big(\bigcap\nolimits_{i}^{\downarrow} P_i\Big) \\ &= \inf \bigcap\nolimits_{i}^{\downarrow} f(P_i) = \bigvee_i^{\uparrow} \inf f(P_i) = \bigvee_i^{\uparrow} \hat{f}(P_i).\end{aligned}$$

Linearity of $\hat{f}$ is straightforward to check using the linearity of f. Finally, regarding binary infima, we have

$$\hat{f}(P \wedge Q) = \inf f(P \wedge Q) = \inf f(\uparrow \mathrm{conv}(P \cup Q)).$$

Now, $\uparrow \mathrm{conv}(P \cup Q)$ is compact saturated and convex, hence its image under the continuous linear function f is compact and convex, too, hence of the form $[r, +\infty]$ or $[r, s[$ or $[r, s]$—so the infimum of $f(\uparrow \mathrm{conv}(P \cup Q))$ is r. Using Proposition 4.8 resp. its equivalent for the classical Smyth powerdomain $\mathcal{S}_c(C)$ as well as the very definition of the convex hull, it follows that r must be one of the infima $\inf f(P)$ and $\inf f(Q)$, namely the smaller one. Hence it is equal to the binary infimum of $\hat{f}(P)$ and $\hat{f}(Q)$, which finishes the proof. □

5 The Convex Hoare Powercone $\mathcal{H}(C)$

In view of partial correctness, we introduce the convex Hoare Powercone. Partial correctness means: if the program terminates, it produces the correct result.

5.1 Definition and Basic Properties

Definition 5.1. For a semitopological cone C, consider

$$\mathcal{H}(C) := \left\{ A \subseteq C \;\middle|\; A \text{ nonempty, convex, closed} \right\}$$

ordered by inclusion $\subseteq$ and equipped with the following operations of addition and scalar multiplication:

$$\begin{aligned} +_{\mathcal{H}}\colon \mathcal{H}(C) \times \mathcal{H}(C) \to \mathcal{H}(C), &\quad A +_{\mathcal{H}} B := \overline{A + B} \\ \cdot_{\mathcal{H}}\colon \mathbb{R}_+ \times \mathcal{H}(C) \to \mathcal{H}(C), &\quad r \cdot_{\mathcal{H}} A := r \cdot A. \end{aligned}$$

On this collection, consider the *upper topology* generated by the following subbasic open sets:

$$(\downarrow B)^{\mathsf{c}} = \mathcal{H}(C) \setminus \downarrow B = \left\{ A \in \mathcal{H}(C) \;\middle|\; A \not\subseteq B \right\} \qquad \text{for } B \in \mathcal{H}(C).$$

Then $\mathcal{H}(C)$ is called the *convex lower powercone* or *convex Hoare powercone* over C.

Theorem 5.2. *For every semitopological cone C, the collection $\mathcal{H}(C)$ with the operations $+_{\mathcal{H}}$ and $\cdot_{\mathcal{H}}$ and the upper topology is a semitopological cone.*

The remainder of this subsection is devoted to the proof of this theorem. First, we verify that the operations $+_{\mathcal{H}}$ and $\cdot_{\mathcal{H}}$ on $\mathcal{H}(C)$ are well-defined:

Proposition 5.3. *The operations of addition*

$$+_{\mathcal{H}}\colon \mathcal{H}(C) \times \mathcal{H}(C) \to \mathcal{H}(C), \quad (A, B) \mapsto \overline{A + B}$$

and scalar multiplication

$$\cdot_{\mathcal{H}}\colon \mathbb{R}_+ \times \mathcal{H}(C) \to \mathcal{H}(C), \quad (r, A) \mapsto r \cdot A$$

on the convex Hoare powercone $\mathcal{H}(C)$ are well-defined for any semitopological cone C.

Proof.
Addition:
Let $A, B \in \mathcal{H}(C)$. Since A and B are both nonempty, so is $A+B$, as well as $A +_{\mathcal{H}} B = \overline{A+B}$, which is furthermore closed by definition. We have that $A+B$ is convex:

Let $x, y \in A+B$ with $x = a_x + b_x$ and $y = a_y + b_y$. For $r \in [0,1]$, we have

$$\begin{aligned} r \cdot x + (1-r) \cdot y &= r \cdot a_x + r \cdot b_x + (1-r) \cdot a_y + (1-r) \cdot b_y \\ &= \underbrace{(r \cdot a_x + (1-r) \cdot a_y)}_{\in A} + \underbrace{(r \cdot b_x + (1-r) \cdot b_y)}_{\in B} \in A+B \end{aligned}$$

Finally, the closure of a convex set is again closed by Lemma 3.7.

Scalar multiplication:
For $r = 0$, we have $r \cdot A = \{0\}$, which is clearly in $\mathcal{H}(C)$. For $r > 0$, the map $a \mapsto r \cdot a$ is a linear homeomorphism on C, hence $r \cdot A$ is convex, closed and nonempty whenever A is. Hence $A \in \mathcal{H}(C)$ implies $r \cdot A \in \mathcal{H}(C)$, which finishes the proof. □

Next, we verify that the operations of addition and scalar multiplication on $\mathcal{H}(C)$ are separately continuous and jointly continuous, respectively. We begin with a lemma:

Lemma 5.4. *In a cone, the preimage of a convex set under an affine function is again a convex set.*

Proof. Let $A \subseteq C$ be a convex subset of a cone, and let $f\colon C \to D$ be an affine function between cones. Let $t \in D$ and $f'\colon C \to D$ be a linear function such that $f = f' + t$.

Take $x, y \in f^{-1}(A)$ in the preimage of A under f. That is, we have $f(x) = f'(x) + t \in A$ and likewise for y. For any $r \in [0,1]$, we have, by linearity of f',

$$\begin{aligned} f\big(r \cdot x + (1-r) \cdot y\big) &= f'\big(r \cdot x + (1-r) \cdot y\big) + t \\ &= r \cdot f'(x) + (1-r) \cdot f'(y) + t \\ &= r \cdot f'(x) + (1-r) \cdot f'(y) + r \cdot t + (1-r) \cdot t \\ &= r \cdot \big(f'(x) + t\big) + (1-r) \cdot \big(f'(y) + t\big) \\ &= r \cdot f(x) + (1-r) \cdot f(y), \end{aligned}$$

which is in A, since $f(x), f(y) \in A$ by assumption and A is a convex set. Hence, we have $r \cdot x + (1-r) \cdot y \in f^{-1}(A)$. □

With this, we are ready to prove separate continuity of addition.

Proposition 5.5. *Let C be a semitopological cone. Then addition on $\mathcal{H}(C)$ as defined by the map*

$$+_{\mathcal{H}}\colon \mathcal{H}(C) \times \mathcal{H}(C) \to \mathcal{H}(C), \quad (A, B) \mapsto \overline{A + B}$$

is separately continuous w.r.t. the upper topology.

Proof. Fix $A \in \mathcal{H}(C)$. We have to show that the function

$$+_{\mathcal{H},A}\colon \mathcal{H}(C) \to \mathcal{H}(C), \quad B \mapsto A +_{\mathcal{H}} B$$

is continuous with respect to the upper topology.

Let $D \in \mathcal{H}(C)$, let $\downarrow D$ be subbasic closed in $\mathcal{H}(C)$. We have to show that the set $+_{\mathcal{H},A}^{-1}(\downarrow D)$ is closed in $\mathcal{H}(C)$.

We have

$$\begin{aligned} +_{\mathcal{H},A}^{-1}\big(\downarrow D\big) &= \big\{\, B \in \mathcal{H}(C) \,\big|\, +_{\mathcal{H},A}(B) \in \downarrow D \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, A +_{\mathcal{H}} B \in \downarrow D \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, A +_{\mathcal{H}} B \subseteq D \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, \overline{A + B} \subseteq D \,\big\} \end{aligned}$$

(since the closure operator is order preserving and D is closed, we have $\overline{A + B} \subseteq \overline{D} = D \iff A + B \subseteq D$)

$$\begin{aligned} &= \big\{\, B \in \mathcal{H}(C) \,\big|\, A + B \subseteq D \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, A \times B \subseteq +^{-1}(D) \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, \forall a \in A:\ \{\, a \,\} \times B \subseteq +^{-1}(D) \,\big\} \end{aligned}$$

(the map $+_a\colon C \to C$, $x \mapsto a+x$ is continuous for any $a \in C$ by assumption, since C is a semitopological cone)

$$\begin{aligned} &= \big\{\, B \in \mathcal{H}(C) \,\big|\, \forall a \in A:\ B \subseteq +_a^{-1}(D) \,\big\} \\ &= \big\{\, B \in \mathcal{H}(C) \,\big|\, B \subseteq \underbrace{\bigcap_{a \in A} +_a^{-1}(D)}_{=:\, D'} \,\big\} \\ &= \downarrow D'. \end{aligned}$$

Since D is a closed and convex subset, so are the continuous affine preimages $+_a^{-1}(D)$ by Lemma 5.4, as well as an arbitrary intersection thereof, such as D'. If D' happens to be empty, so is $\downarrow D'$, and hence it is a closed set. Otherwise, we have $D' \in \mathcal{H}(C)$, and $\downarrow D'$ is a subbasic closed set in the upper topology, which finishes the proof. $\square$

Remark. Note that upper and lower closures $\uparrow$ and $\downarrow$ are always taken within their respective spaces. Hence, in the above proof, $\downarrow D'$ stands for $\{ B \in \mathcal{H}(C) \mid B \subseteq D' \}$ and not for $\{ B \subseteq C \mid B \subseteq D' \}$, so we can safely conclude $D' = \varnothing \implies \downarrow D' = \varnothing$, since the sets in $\mathcal{H}(C)$ are nonempty.

We go on with scalar multiplication.

Proposition 5.6. *Let C be a semitopological cone. Then scalar multiplication on $\mathcal{H}(C)$ as defined by the map*

$$\cdot_{\mathcal{H}} \colon \mathbb{R}_+ \times \mathcal{H}(C) \to \mathcal{H}(C), \quad (r, A) \mapsto r \cdot A$$

is jointly continuous w.r.t. the upper topology.

Proof. According to the remark following Definition 1.23, we only have to check for separate continuity.

Firstly, we have to show that for each fixed scalar $r \in \mathbb{R}_+$, the selfmap $\cdot_{\mathcal{H},r} \colon \mathcal{H}(C) \to \mathcal{H}(C)$, $A \mapsto r \cdot A$ is continuous. For $r = 0$, the map is constant, hence continuous. So let $r > 0$.

Let $D \in \mathcal{H}(C)$, let $\downarrow D$ be subbasic closed. Consider the preimage $\cdot_{\mathcal{H},r}^{-1}(\downarrow D)$ of the scalar multiplication function with fixed scalar r. We have

$$\begin{aligned} \cdot_{\mathcal{H},r}^{-1}(\downarrow D) &= \{ A \in \mathcal{H}(C) \mid \cdot_{\mathcal{H},r}(A) \in \downarrow D \} \\ &= \{ A \in \mathcal{H}(C) \mid r \cdot A \subseteq D \} \\ &= \{ A \in \mathcal{H}(C) \mid A \subseteq \tfrac{1}{r} \cdot D \} \\ &= \downarrow\big(\tfrac{1}{r} \cdot D\big) \end{aligned}$$

Since scalar multiplication defines a linear homeomorphism on C, the subset $\frac{1}{r} \cdot D$ is closed and convex, and it is clearly nonempty, hence an element of $\mathcal{H}(C)$. Hence we have $\downarrow\big(\frac{1}{r} \cdot D\big)$ subbasic closed, which finishes the first part of the proof.

For the second part of the proof, we fix $A \in \mathcal{H}(C)$ and show that the map $\cdot_{\mathcal{H},A} \colon \mathbb{R}_+ \to \mathcal{H}(C)$, $r \mapsto r \cdot A$ is continuous. Consider a subbasic closed set $\downarrow D$ in $\mathcal{H}(C)$. We have

$$\begin{aligned} \cdot_{\mathcal{H},A}^{-1}\big(\downarrow D\big) &= \{ r \in \mathbb{R}_+ \mid \cdot_{\mathcal{H},A}(r) \in \downarrow D \} \\ &= \{ r \in \mathbb{R}_+ \mid r \cdot A \subseteq D \} \\ &= \{ r \in \mathbb{R}_+ \mid \forall a \in A : \ r \cdot a \in D \} \end{aligned}$$

Let $\cdot_a\colon \mathbb{R}_+ \to C$, $r \mapsto r \cdot a$ denote scalar multiplication by a fixed element $a \in C$. By assumption, C is a semitopological cone, hence for any $a \in C$ the function $\cdot_a$ is continuous.

$$\begin{aligned}
&= \left\{\, r \in \mathbb{R}_+ \;\middle|\; \forall a \in A:\; r \in \cdot_a^{-1}(D) \,\right\} \\
&= \left\{\, r \in \mathbb{R}_+ \;\middle|\; r \in \bigcap_{a \in A} \cdot_a^{-1}(D) \,\right\} \\
&= \underbrace{\bigcap_{a \in A} \cdot_a^{-1}(D)}_{=:\, D'}
\end{aligned}$$

Since D is closed, so are the continuous preimages $\cdot_a^{-1}(D)$, as well as an arbitrary intersection thereof, such as D'. Hence, the preimage D' of a subbasic closed set $\downarrow D$ is closed, which finishes the proof. □

Now we are left to verify the cone axioms.

Proposition 5.7. *The triple* $\big(\mathcal{H}(C), +_{\mathcal{H}}, \cdot_{\mathcal{H}}, \{\,0\,\}\big)$ *is a cone.*

Proof. For every $A, B \in \mathcal{H}(C)$, we have $A +_{\mathcal{H}} B = B +_{\mathcal{H}} A$, since addition on C is commutative. Furthermore, we clearly have $A +_{\mathcal{H}} \{\,0\,\} = A$ and $\{\,0\,\} \in \mathcal{H}(C)$. For the associativity of addition, we verify

$$\begin{aligned}
\big(A +_{\mathcal{H}} B\big) +_{\mathcal{H}} C &= \overline{\overline{A + B} + C} \\
&= \overline{(A + B) + C} && \text{by continuity of } + \text{ and Lemma 1.9} \\
&= \overline{A + (B + C)} && \text{by associativity of } + \\
&= \overline{A + \overline{B + C}} && \text{by continuity of } + \text{ and Lemma 1.9} \\
&= A +_{\mathcal{H}} \big(B +_{\mathcal{H}} C\big).
\end{aligned}$$

For scalar multiplication, we only verify one distributivity rule. The other equations are immediate. We show $(r + s) \cdot_{\mathcal{H}} A = (r \cdot_{\mathcal{H}} A) +_{\mathcal{H}} (s \cdot_{\mathcal{H}} A)$:

$$\begin{aligned}
& (r + s)P = rP + sP && \text{by Lemma 4.5} \\
\Longrightarrow\; & \overline{(r + s)P} = \overline{rP + sP} && \text{since } \overline{\,\cdot\,} \text{ is monotone} \\
\Longleftrightarrow\; & (r + s) \cdot_{\mathcal{H}} P = (r \cdot_{\mathcal{H}} P) +_{\mathcal{H}} (s \cdot_{\mathcal{H}} P) && \text{by definition of } +_{\mathcal{H}} \text{ and } \cdot_{\mathcal{H}}
\end{aligned}$$

With this, $\mathcal{H}(C)$ is a cone. □

5.2 The supremum $\vee$: Distributivity laws and continuity

Proposition 5.8. *On the convex Hoare powercone $\mathcal{H}(C)$ over a semitopological cone C, binary suprema $\vee$ exist and are given by*

$$A \vee B = \overline{\operatorname{conv}(A \cup B)}.$$

Proof. By Lemma 3.7, the set $\overline{\operatorname{conv} A \cup B}$ is not only closed but convex. It is nonempty, since $A, B \in \mathcal{H}(C)$ are nonempty by assumption. Hence, it is an element of the convex Hoare powercone and contains both A and B, hence is an upper bound in the order $\subseteq$ of inclusion. On the other hand, any upper bound of A and B in $\mathcal{H}(C)$ would have to be convex and closed, hence would have to comprise the convex hull of their union, as well as the closure thereof. Hence, we do indeed get the smallest upper bound with $\overline{\operatorname{conv}(A \cup B)}$. □

Proposition 5.9. *On the convex Hoare powercone $\mathcal{H}(C)$ over a semitopological cone C, the supremum $\vee$ satisfies the following distributivity laws:*

(i) $A +_{\mathcal{H}} \big(B \vee B'\big) = \big(A +_{\mathcal{H}} B\big) \vee \big(A +_{\mathcal{H}} B'\big)$,

(ii) $r \cdot_{\mathcal{H}} \big(B \vee B'\big) = \big(r \cdot_{\mathcal{H}} B\big) \vee \big(r \cdot_{\mathcal{H}} B'\big)$.

Proof.

(i) We have that $+_{\mathcal{H},A}$ is continuous, hence monotone w.r.t. the order of specialisation, which is just the order $\subseteq$ of inclusion—since the upper topology always reflects the order with respect to which it was created. Hence we immediately get

$$A +_{\mathcal{H}} B = +_{\mathcal{H},A}(B) \subseteq +_{\mathcal{H},A}(B \vee B') = A +_{\mathcal{H}} \big(B \vee B'\big)$$

and likewise

$$A +_{\mathcal{H}} B' = +_{\mathcal{H},A}(B') \subseteq +_{\mathcal{H},A}(B \vee B') = A +_{\mathcal{H}} \big(B \vee B'\big),$$

whence

$$\big(A +_{\mathcal{H}} B\big) \vee \big(A +_{\mathcal{H}} B'\big) \subseteq A +_{\mathcal{H}} \big(B \vee B'\big).$$

Now for the reverse inclusion: We apply Lemma 1.9 to the function $+$ to see that

$$A +_{\mathcal{H}} \big(B \vee B'\big) = \overline{A + \overline{\operatorname{conv}(B \cup B')}} = \overline{A + \operatorname{conv}(B \cup B')}.$$

So it suffices to show $A + \operatorname{conv}(B \cup B') \subseteq (A +_{\mathcal{H}} B) \vee (A +_{\mathcal{H}} B')$. For this purpose, let $x \in A + \operatorname{conv}(B \cup B')$. We have

$$\begin{aligned} x &= a + r \cdot b + (1-r) \cdot b' \\ &= r \cdot (a+b) + (1-r) \cdot (a+b') \\ &\in \operatorname{conv}((A+B) \cup (A+B')) \\ &\subseteq \overline{\operatorname{conv}\Big(\overline{(A+B)} \cup \overline{(A+B')}\Big)} \\ &= \big(A +_{\mathcal{H}} B\big) \vee \big(A +_{\mathcal{H}} B'\big), \end{aligned}$$

which finishes the proof for part (i).

(ii) We begin by a case distinction: If $r = 0$, the claim trivially holds. So let $r > 0$. Again, we use Proposition 1.8 on the map $\cdot_{\mathcal{H},r}$ and the fact that scalar multiplication is a homeomorphism on C to get

$$\begin{aligned} r \cdot_{\mathcal{H}} \big(A \vee B\big) &= r \cdot \overline{\operatorname{conv}(A \vee B)} \\ &= \overline{r \cdot \overline{\operatorname{conv}(A \cup B)}} \\ &= \overline{r \cdot \operatorname{conv}(A \cup B)} \\ &= \overline{\operatorname{conv}\big((r \cdot A) \cup (r \cdot B)\big)} \\ &= \big(r \cdot_{\mathcal{H}} A\big) \vee \big(r \cdot_{\mathcal{H}} B\big), \end{aligned}$$

which finishes the proof for part (ii).

$\square$

We finish the subsection on the supremum by exhibiting its continuity as a function.

Proposition 5.10. *The supremum function on $\mathcal{H}(C)$ defined by*

$$\vee \colon \mathcal{H}(C) \times \mathcal{H}(C) \to \mathcal{H}(C), \qquad (A, B) \mapsto \overline{\operatorname{conv} A \cup B}$$

is jointly continuous.

Proof. Take $\downarrow D \subseteq \mathcal{H}(C)$ subbasic closed. Then the preimage

$$\begin{aligned} \vee^{-1}(\downarrow D) &= \big\{ (A, B) \in \mathcal{H}(C) \times \mathcal{H}(C) \;\big|\; A \vee B = \overline{\operatorname{conv} A \cup B} \subseteq D \big\} \\ &= \big\{ (A, B) \in \mathcal{H}(C) \times \mathcal{H}(C) \;\big|\; A \subseteq D \quad \text{and} \quad B \subseteq D \big\} \\ &= \downarrow D \times \downarrow D \end{aligned}$$

is subbasic closed in $\mathcal{H}(C) \times \mathcal{H}(C)$ with the product of the upper topologies. $\square$

5.3 $\mathcal{H}(C)$ as a stably compact space

Our main result in this subsection is that the convex Hoare powercone construction preserves the property of being stably compact. In other words: If C is a stably compact space, then so is $\mathcal{H}(C)$. As in the case of the Smyth powercone, the proof will involve an additional, similar construction, that of the *classical Hoare powercone* $\mathcal{H}_c(C)$. We will use results from [GHK$^+$03] to examine the classical Hoare powercone and transfer some of its properties to the convex Hoare powercone via a projection.

Some remarks are in order. Our aim is to turn $\mathcal{H}(C)$ with the upper topology into a stably compact space. Here, the order is that of subset inclusion $\subseteq$. In proceeding with the proof, we will occasionally equip our powercones with the *reverse order* $\supseteq$ and the corresponding *lower topology*. Whereas the topologies agree (upper with $\subseteq$ and lower with $\supseteq$), the properties of the spaces as dcpo's differ considerably. We will thus use the reverse order in our proofs to apply the appropriate domain theoretic results from [GHK$^+$03].

In general, since we usually construct our topologies on a space using the order, we will always state the order—even if we deal with the topological properties of the space.

We begin with a definition. For a semitopological cone C, the classical Hoare powercone $\mathcal{H}_c(C)$ contains all nonempty closed subsets of C:

$$\mathcal{H}_c(C) := \left\{ A \subseteq C \;\middle|\; A \text{ nonempty and closed} \right\}.$$

So in comparison to the convex Hoare powercone $\mathcal{H}(C)$, we drop the condition that the subsets be convex. Hence, the classical construction can be applied to any topological space, not only to topological cones. All results about $\mathcal{H}_c(C)$ stated below hold true without the specific cone properties of C. The only reason we do not use simply a topological space, is that we want to connect our results to the convex Hoare powercone $\mathcal{H}(C)$ at the end.

In the sequel, we will equip our given cone C with the patch topology (see Definition 1.17). To indicate this change in topology, we will write C^P.

Theorem 5.11. *Let C be a stably compact semitopological cone. Then $\mathcal{H}_c(C^P)$ with the order $\supseteq$ of reverse subset inclusion and the Lawson topology is a compact Hausdorff space and a complete continuous semilattice.*

Proof. We have that C is stably compact, i.e. compact, coherent, well-filtered and locally compact. We equip C with the patch topology w.r.t. $\subseteq$

and write C^P to indicate this change. Since C is compact, coherent and sober (recall that sober is equivalent to well-filtered in the presence of local compactness), the space C^P is compact by [GHK$^+$03, Lemma VI-6.5, p.475]. Since C is locally compact, C^P is a pospace by [GHK$^+$03, Lemma VI-6.6, p.476] and hence Hausdorff by [GHK$^+$03, Prop. VI-1.4, p.441].

So $\mathcal{H}_c(C^P)$ is the classical Hoare powercone over a compact Hausdorff space. Now, [GHK$^+$03, Example VI-3.8, p.454 and Exercise VI-3.18, p.458] tell us that the classical Hoare powercone over a compact Hausdorff space is a compact Hausdorff space and a complete continuous semilattice when equipped with the order $\supseteq$ of reverse subset inclusion and the Lawson topology. □

Before we return to the convex Hoare powercone, we need a definition and three lemmas.

Definition 5.12. A selfmap $c\colon X \to X$ on a dcpo $(X, \sqsubseteq)$ is called *closure operator* if it is a projection that lies above the identity map, that is, if we have $x \sqsubseteq c(x)$ for any $x \in X$.

Lemma 5.13. *The map*

$$i\colon \mathcal{H}_c(C^P) \to \mathcal{H}_c(C^P) \qquad \textit{with} \qquad A \mapsto \overline{\operatorname{conv} A},$$

where the closure $\overline{\cdot}$ is taken w.r.t. the original topology on C, is a Scott-continuous closure operator w.r.t. the order $\supseteq$ of reverse inclusion.

Proof. First, we show that i is a closure operator:

i **is monotone:** Since both the convex hull operator $\operatorname{conv}(\cdot)$ and the closure operator $\overline{\cdot}$ are monotone, we get:

$$\begin{aligned} A \subseteq B &\implies \operatorname{conv} A \subseteq \operatorname{conv} B \\ &\implies \overline{\operatorname{conv} A} \subseteq \overline{\operatorname{conv} B} \\ &\implies i(A) \subseteq i(B). \end{aligned}$$

i **is idempotent:** We have

$$i \circ i\,(A) \;=\; i\big(i(A)\big) \;=\; \overline{\operatorname{conv}\left(\overline{\operatorname{conv} A}\right)}.$$

By Lemma 3.7, the set $\overline{\operatorname{conv} A}$ is already convex, and thus we get $\operatorname{conv}\left(\overline{\operatorname{conv} A}\right) = \overline{\operatorname{conv} A}$, and hence

$$\overline{\operatorname{conv}\left(\overline{\operatorname{conv} A}\right)} = \overline{\overline{\operatorname{conv} A}} = \overline{\operatorname{conv} A} = i(A),$$

which was to show.

i **lies above the identity:** We have $A \subseteq \operatorname{conv} A \subseteq \overline{\operatorname{conv} A}$ for any $A \subseteq C$, so in particular for any $A \in \mathcal{H}_c(C^P)$.

Finally, we show that i preserves directed suprema: We have that the image $\operatorname{im}(i) = i\big(\mathcal{H}_c(C^P)\big)$ is simply $\mathcal{H}(C)$. Now, $\mathcal{H}(C)$ with the order $\supseteq$ of reverse inclusion is clearly closed under the formation of directed suprema, given by directed intersections—since both convexity and closedness are preserved under the formation of arbitrary intersection, and since a filtered intersection of nonempty closed sets cannot be empty on a compact space. But by [GHK+03, Lemma I-2.4, p.81], a closure operator on a dcpo is Scott-continuous, i.e. preserves directed suprema, if and only if its image is closed under the formation of directed suprema. This finishes the proof. □

Lemma 5.14. *The map $i' \colon \mathcal{H}_c(C^P) \to \mathcal{H}(C)$ with $A \mapsto \overline{\operatorname{conv} A}$ preserves directed suprema and arbitrary infima.*

Proof. The map i' is a restricted version of the selfmap $i\colon \mathcal{H}_c(C^P) \to \mathcal{H}_c(C^P)$ defined in the previous Lemma 5.13. Like i, and by the very same argument, i' preserves directed suprema, given by directed intersection. It does also preserve arbitrary infima, given on $\mathcal{H}_c(C^P)$ by $A \wedge B = \overline{A \cup B}$ and on $\mathcal{H}(C)$ by $A \wedge B = \overline{\operatorname{conv} A \cup B}$. To see this, take any system $(A_i) \subseteq \mathcal{H}_c(C^P)$. We have

$$
\begin{aligned}
i'\Big(\underbrace{\bigwedge_i A_i}_{\wedge \text{ in } \mathcal{H}_c(C^P)}\Big) = i'\Big(\overline{\bigcup_i A_i}\Big) &= \overline{\operatorname{conv} \overline{\bigcup_i A_i}} \\
&\overset{\bigstar}{=} \overline{\operatorname{conv} \bigcup_i A_i} \qquad \text{to be shown} \\
&= \overline{\operatorname{conv} \bigcup_i \overline{\operatorname{conv} A_i}} \\
&= \underbrace{\bigwedge_i i'(A_i)}_{\wedge \text{ in } \mathcal{H}(C)} .
\end{aligned}
$$

If we can show the equality $\bigstar$, we are done. The inclusion "$\supseteq$" is clear. We show the reverse inclusion "$\subseteq$" in more generality by replacing $\bigcup_i A_i$

with some arbitrary subset $X \subseteq C$:

$$
\begin{aligned}
& X \subseteq \operatorname{conv} X \\
\Longrightarrow\quad & \overline{X} \subseteq \overline{\operatorname{conv} X} && \text{since closure } \overline{\cdot} \text{ is monotone} \\
\Longrightarrow\quad & \operatorname{conv} \overline{X} \subseteq \operatorname{conv} \overline{\operatorname{conv} X} && \text{since conv is monotone} \\
\Longrightarrow\quad & \operatorname{conv} \overline{X} \subseteq \overline{\operatorname{conv} X} && \text{since } \overline{\operatorname{conv} X} \text{ is convex by Lemma 3.7} \\
\Longrightarrow\quad & \overline{\operatorname{conv} \overline{X}} \subseteq \overline{\overline{\operatorname{conv} X}} && \text{since closure } \overline{\cdot} \text{ is monotone} \\
\Longrightarrow\quad & \overline{\operatorname{conv} \overline{X}} \subseteq \overline{\operatorname{conv} X} && \text{since closure } \overline{\cdot} \text{ is idempotent}
\end{aligned}
$$

This proves equation ★, and thus finishes the proof. □

Lemma 5.15. *Let C be a stably compact semitopological cone. Then the convex Hoare powercone $\mathcal{H}(C)$, equipped with the order $\supseteq$ of reverse subset inclusion and the Lawson topology, is a compact Hausdorff space.*

Proof. Consider the map $i' \colon \mathcal{H}_c(C^P) \to \mathcal{H}(C)$, $A \mapsto \overline{\operatorname{conv} A}$ from the previous Lemma 5.14. Both its domain and its target space are complete semilattices with the order $\supseteq$ of reverse subset inclusion. By Lemma 5.14, the map i' preserves directed suprema and arbitrary infima. In particular, i' preserves nonempty finite infima, hence is a semilattice homomorphism. By [GHK+03, Theorem III-1.8, p.213], a semilattice homomorphism between complete semilattices preserves directed suprema and nonempty infima if and only if it is a continuous map w.r.t. the Lawson topologies on the domain and target space.

Now, the starting space $(\mathcal{H}_c(C^P), \supseteq)$ is compact in the Lawson topology by Theorem 5.11, and i' is surjective. Hence, the target space $(\mathcal{H}(C), \supseteq)$ with the Lawson topology is the continuous image of a compact space, hence compact, which finishes the proof. □

Finally, we apply our results to the convex Hoare powercone $\mathcal{H}(C)$:

Theorem 5.16. *Let C be a stably compact semitopological cone. Then the Hoare powercone $\mathcal{H}(C)$ with the order $\subseteq$ of subset inclusion and the upper topology is stably compact.*

Proof. By Lemma 5.13, we have that $\mathcal{H}(C)$ is the image of $\mathcal{H}_c(C^P)$ under a closure operator preserving directed suprema. Since $(\mathcal{H}_c(C^P), \supseteq)$ is a continuous semilattice by Theorem 5.11, so is the image $(\mathcal{H}(C), \supseteq)$ by [GHK+03, Corollary I-2.3, p.81]. Furthermore, $\mathcal{H}(C)$ is Lawson-compact by Lemma 5.15. By [GHK+03, Prop. VI-6.24, p.482], a continuous Lawson-compact semilattice is stably compact both with the lower and with the

Scott topology. Finally, the lower topology on $(\mathcal{H}(C), \supseteq)$ is equal to the upper topology on $(\mathcal{H}(C), \subseteq)$. $\square$

5.4 A property for the composition of programs

We finish the section about the convex Hoare powercone by giving an analogon of Section 4.2 for the Smyth powercone.

Lemma 5.17. *Let X be a topological T_0-space with specialisation order $\sqsubseteq$. Then for every subset $A \subseteq X$, we have $\bigvee \overline{A} = \bigvee A$.*

Proof. We show that every subset A has the same upper bounds as its closure $\overline{A}$. The least upper bound must then be the same.

Clearly, since $A \subseteq \overline{A}$, every upper bound of $\overline{A}$ is also an upper bound of A. So let $x \in X$ be an upper bound for A. Let $a \in \overline{A} \setminus A$, and let $(a_i) \subseteq A$ be a net converging to a. We have $a_i \sqsubseteq x$ for every i, because the net is in A. Since the net converges to a, in every open neighbourhood $U \in \mathcal{N}(a)$ of a we find some a_i. Since x lies above all a_i and open sets are upper sets, this means that x is contained in every open neighbourhood of a. So we have $\mathcal{N}(a) \subseteq \mathcal{N}(x)$, which is just the definition of $a \sqsubseteq x$. Thus, x is an upper bound for A and for all $a \in \overline{A} \setminus A$, hence an upper bound for $\overline{A}$, which finishes the proof. $\square$

Proposition 5.18. *Let C, D be stably compact semitopological cones, where D has binary suprema $\vee$ with $+$ and $\cdot$ distributing over $\vee$. Let $f \colon C \to D$ be a continuous linear map. Then the map defined by*

$$\hat{f} \colon \mathcal{H}(C) \to D \quad \textit{with} \quad \hat{f}(A) := \bigvee f(A)$$

is linear.

Proof. Let $+_{\mathcal{H}}$ and $\cdot_{\mathcal{H}}$ denote addition and scalar multiplication on $\mathcal{H}(C)$, let $A, B \in \mathcal{H}(C)$.

We verify that $\hat{f}$ preserves addition:

$$\begin{aligned}
\hat{f}\big(A +_{\mathcal{H}} B\big) &= \bigvee f\big(A +_{\mathcal{H}} B\big) \\
&= \bigvee f\big(\overline{A+B}\big) \\
&= \bigvee \overline{f\,\big(\overline{A+B}\big)} && \text{by Lemma 5.17} \\
&= \bigvee \overline{f\,(A+B)} && \text{by continuity of } f \text{ and Proposition 1.8} \\
&= \bigvee \overline{f(A)+f(B)} && \text{by linearity of } f \\
&= \bigvee f(A)+f(B) && \text{by Lemma 5.17} \\
&= \bigvee f(A)+\bigvee f(B) && \text{as } + \text{ distributes over } \vee \text{ in } D \\
&= \hat{f}(A)+\hat{f}(B).
\end{aligned}$$

By an analogous proof, we get $\hat{f}(r \cdot_{\mathcal{H}} A) = \bigvee r \cdot f(A)$, and since scalar multiplication defines an order isomorphism on D, this is in turn equal to $r \cdot \bigvee f(A) = r \cdot \hat{f}(A)$. Hence $\hat{f}$ is linear. □

Proposition 5.19. *Let C, D be stably compact semitopological cones, where D has binary suprema $\vee$ and is equipped with the upper topology. Let $f\colon C \to D$ be a continuous linear map. Then the map defined by*

$$\hat{f}\colon \mathcal{H}(C) \to D \quad \textit{with} \quad \hat{f}(A) := \bigvee f(A)$$

is continuous.

Proof. Take $\downarrow x \subseteq D$ subbasic closed. We have

$$\begin{aligned}
\hat{f}^{-1}\big(\downarrow x\big) &= \big\{\, A \in \mathcal{H}(C) \bigm| \hat{f}(A) \in \downarrow x \,\big\} \\
&= \big\{\, A \in \mathcal{H}(C) \bigm| \bigvee f(A) \in \downarrow x \,\big\} \\
&= \big\{\, A \in \mathcal{H}(C) \bigm| \bigvee f(A) \sqsubseteq x \,\big\} \\
&= \big\{\, A \in \mathcal{H}(C) \bigm| f(A) \subseteq \downarrow x \,\big\} \\
&= \big\{\, A \in \mathcal{H}(C) \bigm| A \subseteq f^{-1}(\downarrow x) \,\big\} \\
&= \downarrow f^{-1}\big(\downarrow x\big).
\end{aligned}$$

The set $\downarrow x$ is nonempty, upper-closed and convex. Hence the set $f^{-1}(\downarrow x)$ is convex by linearity of f and Lemma 5.4; it is closed by continuity of f; and it is nonempty, since $f(0) = 0$ by linearity of f, and thus

$$f(0) = 0 \sqsubseteq x \iff 0 \in f^{-1}(\downarrow x).$$

Hence, we have $f^{-1}(\downarrow x) \in \mathcal{H}(C)$, whence $\downarrow f^{-1}(\downarrow x)$ is a subbasic closed set in $\mathcal{H}(C)$ with the upper topology. □

Lemma 5.20. *If a function preserves binary suprema and directed suprema, then it preserves arbitrary suprema.*

Proof. Take an arbitrary nonempty subset D of the space. Form the subset

$$D_1 = \left\{ a \vee b \;\middle|\; a, b \in D \right\}$$

of all binary suprema in D. Next, form the subset

$$D_2 = \left\{ a \vee b \;\middle|\; a, b \in D_1 \right\}$$

of all binary suprema in D_1, and so on. Collecting all these binary suprema gives a directed set $D' = \bigcup_{i=1}^{\infty} D_i$. If the directed supremum $\bigvee^{\uparrow} D'$ exists, it is equal to the supremum $\bigvee D$ of D. Any function preserving directed suprema and binary suprema will preserve the supremum of D', and thus the supremum of D. If D was empty, the supremum is the smallest element of the starting space; any function preserving binary suprema also preserves the order, and hence maps the smallest element of the starting space to the smallest element of the target space. Thus, functions preserving directed and binary suprema do indeed preserve arbitrary (even empty) suprema. □

Proposition 5.21. *Let C, D be stably compact semitopological cones, where D has binary suprema $\vee$ with $+$ and $\cdot$ distributing over $\vee$, and D is equipped with the upper topology. Let $f\colon C \to D$ be a continuous linear map. Then the map defined by*

$$\hat{f}\colon \mathcal{H}(C) \to D \quad \textit{with} \quad \hat{f}(A) := \bigvee f(A)$$

is the unique linear continuous map preserving binary suprema that makes the following diagram commute:

$$\begin{tikzcd}
C \arrow[r, "x \mapsto \downarrow x"] \arrow[dr, "f"'] & \mathcal{H}(C) \arrow[d, "\hat{f}"] \\
 & D
\end{tikzcd}$$

Proof.
Continuity, linearity, extension property:
The map $\hat{f}$ is linear and continuous by Propositions 5.18 and 5.19. Since $x = \bigvee \downarrow x = \max \downarrow x$ and continuous maps preserve the specialisation order, we have

$$\hat{f}(\downarrow x) = \bigvee f(\downarrow x) = \max f(\downarrow x) = f(x),$$

whence the diagram commutes.

Preserving binary suprema:
We verify that the extension function $\hat{f}$ preserves binary suprema. We have:

$$\begin{aligned}
\hat{f}\big(A \vee B\big) &= \hat{f}\big(\overline{\operatorname{conv} A \cup B}\big) && \\
&= \bigvee f\big(\overline{\operatorname{conv} A \cup B}\big) && \text{by definition of } \hat{f} \\
&= \bigvee \overline{f\big(\overline{\operatorname{conv} A \cup B}\big)} && \text{by Lemma 5.17} \\
&= \bigvee \overline{f\big(\operatorname{conv} A \cup B\big)} && \text{by Lemma 1.8}
\end{aligned}$$

It is an easy exercise to verify that $f(\operatorname{conv} E) = \operatorname{conv} f(E)$ holds for any linear function f and any subset E.

$$= \bigvee \overline{\operatorname{conv} f(A \cup B)}$$

Taking Lemma 5.17 one step further, we have that $\bigvee E = \bigvee \overline{\operatorname{conv} E}$ for any subset E. (Hint: An element x is an upper bound for E iff $E \subseteq \downarrow x$ iff $\overline{\operatorname{conv} E} \subseteq \downarrow x$, since $\downarrow x$ is already closed and convex.) This gives:

$$\begin{aligned}
&= \bigvee f(A \cup B) \\
&= \bigvee \big(f(A) \cup f(B)\big) \\
&= \bigvee f(A) \vee \bigvee f(B) \\
&= \hat{f}(A) \vee \hat{f}(B),
\end{aligned}$$

which was to show.

Uniqueness:
For the diagram to commute, we have to set $\hat{f}(\downarrow x) := x$. By [GHK$^+$03, Exercise O-5.15(viii), p.46], we have that $\hat{f}$ preserves directed suprema, since it is a continuous function between sober spaces. By assumption, we also

want $\hat{f}$ to preserve binary suprema. Hence, by Lemma 5.20, the function $\hat{f}$ preserves arbitrary suprema.

For any $A \in \mathcal{H}(C)$, we have:

$$\overline{\mathrm{conv}\, A} = A = \bigcup_{x \in A} \downarrow x = \overline{\mathrm{conv} \bigcup_{x \in A} \downarrow x} = \bigvee_{x \in A} \downarrow x = \bigvee_{x \in A}^{\uparrow} \downarrow x.$$

Hence, we obtain

$$\hat{f}(A) = \hat{f}\Big(\bigvee_{x \in A}^{\uparrow} \downarrow x \Big) = \bigvee_{x \in A}^{\uparrow} \hat{f}(\downarrow x) = \bigvee_{x \in A}^{\uparrow} f(x) = \bigvee^{\uparrow} f(A),$$

whence we have no choice for the definition of $\hat{f}$, which shows uniqueness. □

To end this section, consider the category **SemiTop** of stably compact semitopological cones with continuous linear maps as morphisms, as well as the category **SemiTop**$^{\wedge}$ of stably compact semitopological cones with binary suprema, together with continuous linear functions preserving binary suprema as morphisms.

Proposition 5.22. *The assignment $C \mapsto \mathcal{H}(C)$ can be extended to a functor $\mathcal{H} \colon$ **SemiTop** $\to$ **SemiTop**$^{\wedge}$ by assigning to any continuous linear map $f \colon C \to D$ the map*

$$\mathcal{H}(f) \colon \mathcal{H}(C) \to \mathcal{H}(D) \quad \textit{with} \quad A \mapsto \overline{f(A)}$$

which is continuous and linear and preserves binary suprema.

Proof. We have to verify that $\mathcal{H}(f)$ is continuous and linear and that it preserves binary suprema. These are straightforward calculations, using the continuity and linearity of f as well as Lemma 1.9 and Proposition 1.8. We leave them as exercise to the reader.

Finally, for $A \in \mathcal{H}(C)$ we see that

$$\mathcal{H}(\mathrm{id}_C)(A) = \overline{\mathrm{id}_c(A)} = \overline{A} = A = \mathrm{id}_{\mathcal{H}(C)}(A),$$

and for composition we use continuity of f and g together with Proposition 1.8 to get:

$$\mathcal{H}\Big(g \circ f\Big)(A) = \overline{g\big(f(A)\big)} = \overline{g\big(\overline{f(A)}\big)} = \Big(\mathcal{H}(g) \circ \mathcal{H}(f)\Big)(A).$$

□

Conclusion

To conclude this diploma thesis, we give a brief overview of what we have done and of questions arising from our work.

Summary

We have verified that the extended probabilistic powerdomain construction $\mathcal{V}$ can be extended to an endofunctor, and even to a monad in the category $\mathbf{StCp}$ of stably compact spaces. We have verified that separation theorems formerly shown for dcpo-cones still hold with locally convex topological cones. We have shown that the convex Smyth powercone construction $\mathcal{S}$ is a functor from the category $\mathbf{StCpCvCone}$ of stably compact locally convex topological cones into the category $\mathbf{StCpCvCone}^{\wedge}$ of stably compact locally convex topological cones with binary infima. For stably compact locally convex topological cones C, we have exhibited an alternative characterisation of the Scott topology on $\mathcal{S}(C)$, and we have shown that the latter is a continuous dcpo-cone in this case. Finally, we have shown that the convex Hoare powercone construction is a functor from the category $\mathbf{SemiTop}$ of stably compact semitopological cones to the category $\mathbf{SemiTop}^{\vee}$ of stably compact semitopological cones with binary suprema. For all three constructions, $\mathcal{V}$, $\mathcal{S}$ and $\mathcal{H}$, we have proved an extension property that can be used for the denotational semantics of the composition of programs.

Open Questions

An interesting task would be to determine the algebras of the monads $\mathcal{V}$, discussed in Section 2 for the extended probabilistic powerdomain, and $\mathcal{S}$, discussed in Section 4 for the convex Smyth powercone. This would provide a genuine universal property. Looking at the special case of Proposition 2.8, we see that the images of $\mu \in \mathcal{V}(X)$ under the extension function must be uniquely determined by the values on the point valuations η_x, in order for the universal property to hold.

Unfortunately, there is no hope of taking over the procedure used for continuous domains: There, arbitrary continuous valuations are approximated by simple valuations (that is, linear combinations of point valuations) via directed suprema; any function for which we search an extension is a continuous function between sober spaces, hence preserves directed suprema, hence is determined by the images of the point valuations. Kirch has shown in [Kir93, Satz 5.4, p.47] that whenever such an approximation by directed suprema is possible in $\mathcal{V}(X)$, the space X is already a continuous domain (under the assumption that X is sober, which is the case for stably compact spaces). Hence one must find another, more topological way to approximate continuous valuations by point valuations.

Concerning the convex Hoare powercone $\mathcal{H}(C)$, it is still open whether it can be made into a topological, rather than a semitopological cone, provided that the given cone C is a topological cone.

Finally, one could further investigate Proposition 5.19, which states that the extension function on $\mathcal{H}(C)$ is continuous. More precisely, it is unclear whether the target space has to carry the upper topology. Closer examination might show if this requirement is superfluous or necessary.

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