

A Class of Generalised Quantifiers for k-Variable Logics

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Abstract

We introduce k -quantifier logics – logics with access to k -tuples of elements and very general quantification patterns for transitions between k -tuples. The framework is very expressive and encompasses e.g. the k -variable fragments of first-order logic, modal logic, and monotone neighbourhood semantics. We introduce a corresponding notion of bisimulation and prove variants of the classical Ehrenfeucht–Fraïssé and Hennessy–Milner theorem. Finally, we show a Lindström-style characterisation for k -quantifier logics that satisfy Łoś’ theorem by proving that they are the unique maximally expressive logics that satisfy Łoś’ theorem and are invariant under the associated bisimulation relations.

1 Introduction

Lindström’s theorem for first-order logic FO is a hallmark of abstract model theory. In its main version, as treated as Lindström’s First Theorem e.g. in [7], it characterises FO as *maximally expressive* among logics that satisfy some very basic standard properties of abstract logics together with *compactness* and the downward *Löwenheim-Skolem Theorem* as the essential constraints. Among the standard properties that competitors must satisfy are basic semantic principles like isomorphism invariance, closure under booleans, natural behaviour w.r.t. renaming of symbols, and the like. The two crucial extra properties that are characteristic of FO against this backdrop are its compactness (satisfiability of every finite subset implies satisfiability) and the countable Löwenheim-Skolem condition (satisfiability of a countable set of formulae implies satisfiability over an at most countably infinite domain). For FO, compactness is an immediate corollary to Gödel’s Completeness Theorem, but can also be derived – in purely model-theoretic terms – from compatibility of FO with ultraproducts. This concerns the universal-algebraic generation of natural quotients of direct products w.r.t. ultrafilter equivalence. That essential model-theoretic feature of FO is the content of Łoś’ Theorem (cf. Theorem 6.2 below, and see e.g. [7] for a textbook account), which states that the formulae $\varphi \in \text{FO}$ satisfied in such an ultrafilter-reduced product over an infinite family of structures are precisely those for which the set of component structures where φ is satisfied is in the ultrafilter (see Section 6 for a very short review of the basic terminology, which is standard textbook material). Deeper set-theoretic principles are

involved in the Keisler–Shelah Theorem [2, Theorem 6.1.15], which links this Łoś-phenomenon more directly to FO expressiveness by stating that elementarily equivalent structures possess isomorphic ultrapowers. This implies that no logic that respects isomorphisms and is compatible with ultraproducts (or even just ultrapowers, as a special case) can possibly distinguish any two structures that are indistinguishable in FO; in other words: any such logic must not only satisfy compactness, but its expressive power is directly dominated by that of FO. Building on this insight, Sgro established the maximality of logics on certain classes of models as being maximally expressive among logics satisfying Łoś’ theorem [23]. So much for the classical picture.

A much more recent rekindling of the interest in Lindström-like results has come with an emphasis on additional semantic constraints, mostly in the sense of invariance conditions. These are often motivated by typical application domains where often the desirable levels of expressiveness can be considerably weaker than, or possibly also incomparable with, FO. One of the prominent scenarios in this vein is that of modal logics where specific constraints on the accessibility of information are modelled in Kripke structures. Under this modelling regime, the full expressive power of FO would potentially capture irrelevant features while maybe unnecessarily missing out on others. In the basic modal scenario, the natural apriori intended restriction for the semantics is captured by *bisimulation invariance*. This semantic constraint can be seen as a severe strengthening of isomorphism invariance, which however may arguably be no less natural in specific contexts. The renewed investigation of Lindström phenomena with a systematic focus on such restricted scenarios is highlighted in the seminal paper by van Benthem, ten Cate and Väänänen [24], and also explored e.g. in [20]. Clearly Lindström-type maximality assertions admit many variations in terms of the choice of background principles from abstract model theory (like compactness, the Tarski union property TUP, a Łoś-property, or locality principles) besides the additional, domain-specific semantic constraint embodied in invariance conditions or restrictions to classes of models as in frame conditions for modal logic [8]. Depending on how much the background principles already on their own restrict the expressive power of the target logic to a standard logic like FO, a Lindström result may also be read as a *characterisation theorem* in relation to the extra invariance condition. To give one example in the classical modal scenario targeting basic modal logic ML, van Benthem’s characterisation of ML as expressively complete for the bisimulation-invariant fragment of FO can also be cast as a corollary to a Lindström theorem that establishes ML as maximally expressive among bisimulation-invariant logics satisfying some sufficiently strong background conditions that in effect tie them to within FO, cf. [24, 20, 8]. Interestingly, semantic invariance conditions based on Ehrenfeucht–Fraïssé-style back-and-forth equivalences (viz. partial isomorphy) can also be seen at the root of the classical Lindström result (cf. presentation in [7]) and variations for more expressive logics in the spirit of abstract model theory [9]. Here we use a natural adaptation of bisimulation invariance on k -tuples as a semantic invariance condition that directly reflects the semantic constraints in the underlying k -quantifiers. Also in this respect our results highlight – once again – the intimate connection between Lindström and expressive completeness results. In a different vein, recent investigations by Liao [15] generalise the classical Lindström proof to an entire class of logics that can encode their respective Ehrenfeucht–Fraïssé games.

The idea for the present paper originated from the first author’s bachelor’s thesis [14]. We use the Łoś-property of compatibility with ultraproducts as an extremely strong background condition from abstract model theory for a Lindström-style characterisation. While this may look like a possible overkill in light of the Keisler–Shelah Theorem (as it imposes a first-order ceiling for expressiveness), it stands out as a very neat principle that may be motivated from a purely universal-algebraic point of view. But more importantly, our main Lindström argument in Section 7 does not rely on Keisler–Shelah but rather establishes maximality, for a very wide class of target logics in question, by techniques that do not directly draw on their first-order nature – even though that is indirectly imposed by the Łoś-property. And even in light of that deeper implication, which we discuss in Section 6, and for a corresponding interpretation of our main result in terms of characterisation theorems for fragments of FO, the generality w.r.t. the target logics involved and their natural semantic invariance conditions make these results interesting in our view. As for those target logics we focus on a rather general framework that does not immediately suggest first-order character but rather focuses on an intuition of limited access to, and information links between, local configurations or states, as exemplified in familiar examples in the range of modal logics and neighbourhood semantics [21]. For relevant *local configurations* we concentrate on fixed-arity tuples of elements, and correspondingly consider structures with k -tuples of elements, *k -pointed structures*, for fixed $k \geq 1$, as input for the semantics of our abstract logics.¹ The expressive power of our target logics then rests on an a priori very general notion of *k -quantifiers*, which access sets of accessible k -tuples from any given k -tuple in such a structure; the underlying access pattern and style of quantification based on these is very much akin to (monotone) neighbourhood semantics if we think of the sets of accessible k -tuples as neighbourhoods. The main point lies in the extremely liberal format for these k -quantifiers (or the associated neighbourhood systems of witness sets supporting their semantics) that admits almost any sensible (viz. isomorphism invariant) allocation of systems of witness sets in k -pointed structures, cf. Section 2 for details. The additional semantic invariance conditions that come with a collection of k -quantifiers are modelled on the natural generalisations of bisimulation invariance, for a notion of Ehrenfeucht–Fraïssé style back&forth games in which the alternating probing of k -configurations is guided by access to the available witness sets underlying the semantics of k -quantifiers, as discussed in Sections 3, 4 and 5.

Other frameworks for capturing the semantics of entire classes of logics have been explored in the past. Most notably, *Lindström quantifiers* (cf. [5, Chapter 4]) are much more general and versatile than our k -quantifiers. While this expressive strength allows them to capture almost any conceivable logic, their power is rather an obstacle in our setting. Their very generality makes it hard to capture and restrict their expressive power in the manner proposed here.

Neighbourhood semantics for modal logic, as explored extensively in [21], is closely related to our notion of 1-quantifier logics, the unary version of k -quantifier logic. There is essentially a one-to-one correspondence between upward-monotone neighbourhood functions for the box operator and our witness

¹This is similar in spirit to the limitation to k -variable fragments of standard logics; cf. k -variable fragments like $\text{FO}^k \subseteq \text{FO}$ or its infinitary variants, and, e.g. especially their extensions by counting quantifiers, like $C^k \subseteq \text{FO}$, with their established relevance in algorithmic model theory [6, 16, 4, 19].

sets for 1-quantifiers. The two approaches mainly differ in that neighbourhood semantics considers the neighbourhood function as part of the admissible specifications of the structures under consideration, while the witness sets of 1-quantifiers are considered as ‘externally determined’ by the quantifier. One could therefore think of 1-quantifiers as imposing extremely restrictive frame conditions for neighbourhood functions.

For an illustration of the expressive power of 1-quantifier logics see Examples 2.8, 2.9, and 2.10 below.

This paper is structured as follows: In Section 2, we introduce our notion of k -quantifiers and k -quantifier logics, and give well-known examples of logics covered by our framework. We define a corresponding bisimulation game in Section 3. In Sections 4 and 5 we adapt the classical Ehrenfeucht–Fraïssé and Hennessy–Milner theorems to k -quantifier logics. In Section 6, we investigate properties of k -quantifier logics that have the Łoś property and conclude our investigations with a Lindström-style theorem for such k -quantifier logics in Section 7.

2 Abstract k -Logics

Throughout our investigations, we restrict ourselves to purely relational signatures. The arity of a given relation R is denoted as $\text{ar}(R)$.

The universe of a σ -structure \mathfrak{A} is denoted as A . A pair (\mathfrak{A}, α) consisting of a σ -structure \mathfrak{A} and a corresponding variable assignment α is called a *pointed σ -structure* or *pointed structure* for short.

For $k \in \mathbb{N}$, a *k -assignment* is an assignment over k variables. A pointed σ -structure (\mathfrak{A}, α) is a *k -pointed (σ -)structure* if α is a k -assignment. We associate k -assignments $\alpha : \{x_1, \dots, x_k\} \rightarrow A$ with the k -tuple $(\alpha(x_1), \dots, \alpha(x_k)) \in A^k$ to ease notation.

Definition 2.1 (abstract logics). An *abstract logic* $\mathcal{L} = (L, \models_{\mathcal{L}})$ is a pair consisting of

- a function L mapping signatures to sets of formulae, and
- a satisfaction relation $\models_{\mathcal{L}}$ relating pointed σ -structures and formulae in $L(\sigma)$.

We require an abstract logic to

- be invariant under isomorphism, i.e. for any $\varphi \in L(\sigma)$, pair of isomorphic σ -structures $\iota : \mathfrak{A} \simeq \mathfrak{B}$ and variable assignment α in \mathfrak{A} ,

$$\mathfrak{A}, \alpha \models_{\mathcal{L}} \varphi \iff \mathfrak{B}, \iota \circ \alpha \models_{\mathcal{L}} \varphi,$$

- be closed under boolean connectives, i.e.

$$\varphi, \psi \in L(\sigma) \implies \top, \neg\varphi, (\varphi \wedge \psi) \in L(\sigma),$$

- have the (syntactic) finite occurrence property, i.e. for every signature σ , $\varphi \in \mathcal{L}(\sigma)$ there is a finite $\sigma_0 \subseteq \sigma$ s.t. $\varphi \in \mathcal{L}(\sigma_0)$,

- be invariant under renaming, i.e. for any signature σ , fresh k -ary relations $R, R' \notin \sigma$, and $\varphi \in \mathcal{L}(\sigma \cup \{R\})$, there is some $\varphi' \in \mathcal{L}(\sigma \cup \{R'\})$ such that for all pointed σ -structures (\mathfrak{A}, α) and $R^{\mathfrak{A}} = R'^{\mathfrak{A}} \subseteq A^k$,

$$\mathfrak{A}, R^{\mathfrak{A}}, \alpha \models \varphi \iff \mathfrak{A}, R'^{\mathfrak{A}}, \alpha \models \varphi'.$$

An *abstract k -logic* \mathcal{L} is an abstract logic that defines semantics for k -pointed σ -structures with variable assignments over $\mathcal{X}_k := \{x_1, \dots, x_k\}$.

We treat the connectives \vee, \rightarrow and \leftrightarrow as well as the formula \perp as abbreviations in the usual manner. To ease notation, we furthermore do not distinguish between L and \mathcal{L} , or write just \models instead of $\models_{\mathcal{L}}$, where this causes no confusion.

Note that as we mostly consider compact logics throughout our investigations, the finite occurrence property is mainly stated for convenience as compactness and invariance under renaming implies the finite occurrence property in a semantic sense (cf. [5, Proposition 5.1.2]).

An abstract logic \mathcal{L}_1 is a *fragment* of an abstract logic \mathcal{L}_2 or \mathcal{L}_2 an *extension* of \mathcal{L}_1 , denoted as $\mathcal{L}_1 \preceq \mathcal{L}_2$ or $\mathcal{L}_2 \succeq \mathcal{L}_1$, if for every signature σ and every formula $\varphi \in \mathcal{L}_1(\sigma)$, there exists a formula $\psi \in \mathcal{L}_2(\sigma)$ such that for all pointed σ -structures (\mathfrak{A}, α) ,

$$\mathfrak{A}, \alpha \models \varphi \iff \mathfrak{A}, \alpha \models \psi.$$

As syntactic differences are immaterial for our purposes, we assume that for all signatures σ ,

$$\mathcal{L}_1 \preceq \mathcal{L}_2 \implies L_1(\sigma) \subseteq L_2(\sigma).$$

Two abstract logics \mathcal{L}_1 and \mathcal{L}_2 are *equivalent*, denoted as $\mathcal{L}_1 \equiv \mathcal{L}_2$, if $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \preceq \mathcal{L}_1$.

Definition 2.2 (*k -quantifiers*). A *k -quantifier* Q is a symbol together with

- an associated signature σ_Q , and
- a *witness set* $Q(\mathfrak{A}, \alpha) \subseteq \mathcal{P}(A^k)$ for every (expansion of a) k -pointed σ_Q -structure (\mathfrak{A}, α) . Q needs to be well-behaved w.r.t. isomorphisms and expansions, i.e. if $\iota : (\mathfrak{A} \upharpoonright \sigma_Q) \simeq (\mathfrak{B} \upharpoonright \sigma_Q)$, then for any k -assignment α

$$Q(\mathfrak{B} \upharpoonright \sigma_Q, \iota \circ \alpha) = Q(\mathfrak{B}, \iota \circ \alpha) = \{\{\iota \circ \gamma \mid \gamma \in s\} \mid s \in Q(\mathfrak{A} \upharpoonright \sigma_Q, \alpha)\}.$$

Note that the provision that k -quantifiers are well-behaved w.r.t. expansions already implies that k -quantifiers respect isomorphisms.

We call the members of a witness set *witnesses* and, as the underlying structure should usually be clear from context, we may shorten $Q(\mathfrak{A}, \alpha)$ to just $Q(\alpha)$.

For a given set \mathcal{Q} of k -quantifiers, we let

$$\mathcal{Q}_\sigma := \{Q \in \mathcal{Q} \mid \sigma_Q \subseteq \sigma\}.$$

Definition 2.3 (*k -quantifier logics*). A *k -quantifier logic* $\mathcal{L}^{\mathcal{Q}}$ is an abstract k -logic based on a class of k -quantifiers \mathcal{Q} with syntax and semantics given as follows.

For a given signature σ , the *syntax* of $\mathcal{L}^{\mathcal{Q}}(\sigma)$ is given inductively via

- $\top \in \mathcal{L}^{\mathcal{Q}}(\sigma)$,
- for all l -ary relations $R \in \sigma$ and variables $y_1, \dots, y_l \in \mathcal{X}_k$, the formula $R(y_1, \dots, y_l)$ is in $\mathcal{L}^{\mathcal{Q}}(\sigma)$,
- for all $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$, also $\neg\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$,
- for all $\varphi_1, \varphi_2 \in \mathcal{L}^{\mathcal{Q}}(\sigma)$, also $(\varphi_1 \wedge \varphi_2) \in \mathcal{L}^{\mathcal{Q}}(\sigma)$,
- for all $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$ and $Q \in \mathcal{Q}_\sigma$, also $Q\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$.

The *satisfaction relation* is also defined in an inductive fashion: for all signatures σ , $Q \in \mathcal{Q}_\sigma$ and k -pointed σ -structures (\mathfrak{A}, α) ,

- $\mathfrak{A}, \alpha \models \top$,
- $\mathfrak{A}, \alpha \models R(y_1, \dots, y_l)$ if $(\alpha(y_1), \dots, \alpha(y_l)) \in R^{\mathfrak{A}}$,
- $\mathfrak{A}, \alpha \models \neg\varphi$ if $\mathfrak{A}, \alpha \not\models \varphi$,
- $\mathfrak{A}, \alpha \models (\varphi_1 \wedge \varphi_2)$ if $\mathfrak{A}, \alpha \models \varphi_1$ and $\mathfrak{A}, \alpha \models \varphi_2$,
- $\mathfrak{A}, \alpha \models Q\varphi$ if there exists an $s \in Q(\mathfrak{A}, \alpha)$ such that for all $\gamma \in s$: $\mathfrak{A}, \gamma \models \varphi$.

Inspired by flat team semantics, we define for subsets $s \subseteq A^k$ that

$$\mathfrak{A}, s \models \varphi \quad :\iff \quad \text{f.a. } \alpha \in s : \mathfrak{A}, \alpha \models \varphi.$$

Note that while we could include equality as a literal in the definition of k -quantifier logics, we chose not to as it can be easily added by introducing it as an extra binary relation if desired.

Observation 2.4 (upward monotonicity). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic. Then $\mathcal{L}^{\mathcal{Q}}$ is upward monotone, i.e., for any formula $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$, k -pointed σ -structure (\mathfrak{A}, α) , and $s \in Q(\alpha)^\uparrow := \{s' \subseteq A^k \mid \text{ex. } t \in Q(\alpha) \text{ with } t \subseteq s'\}$,*

$$\mathfrak{A}, s \models \varphi \quad \implies \quad \mathfrak{A}, \alpha \models Q\varphi.$$

Definition 2.5 (quantifier rank). For a given k -quantifier logic $\mathcal{L}^{\mathcal{Q}}$ and signature σ , we define the *quantifier rank* of formulae in $\mathcal{L}^{\mathcal{Q}}(\sigma)$ inductively as

$$\begin{aligned} \text{qr}(\top) &:= 0, \\ \text{qr}(R(y_1, \dots, y_l)) &:= 0 && \text{for any } R \in \sigma, \text{ variables } y_1, \dots, y_l, \\ \text{qr}(\neg\varphi) &:= \text{qr}(\varphi) && \text{for any } \varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma), \\ \text{qr}(\varphi_1 \wedge \varphi_2) &:= \max\{\text{qr}(\varphi_1), \text{qr}(\varphi_2)\} && \text{for any } \varphi_1, \varphi_2 \in \mathcal{L}^{\mathcal{Q}}(\sigma), \\ \text{qr}(Q\varphi) &:= 1 + \text{qr}(\varphi) && \text{for any } \varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma), Q \in \mathcal{Q}_\sigma. \end{aligned}$$

Definition 2.6 ($\mathcal{L}^{\mathcal{Q}}$ -elementary equivalence). Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic. Then two k -pointed σ -structures (\mathfrak{A}, α) and (\mathfrak{B}, β) are $\mathcal{L}^{\mathcal{Q}}$ -*elementarily equivalent*, denoted as $(\mathfrak{A}, \alpha) \equiv_{\mathcal{L}^{\mathcal{Q}}} (\mathfrak{B}, \beta)$, if for all $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$,

$$\mathfrak{A}, \alpha \models \varphi \quad \iff \quad \mathfrak{B}, \beta \models \varphi.$$

Similarly, (\mathfrak{A}, α) and (\mathfrak{B}, β) are $\mathcal{L}_q^{\mathcal{Q}}$ -*elementarily equivalent* for $q \in \mathbb{N}$, denoted $(\mathfrak{A}, \alpha) \equiv_{\mathcal{L}_q^{\mathcal{Q}}}^q (\mathfrak{B}, \beta)$, if for all $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$ with $\text{qr}(\varphi) \leq q$,

$$\mathfrak{A}, \alpha \models \varphi \quad \iff \quad \mathfrak{B}, \beta \models \varphi.$$

Definition 2.7 (\mathcal{L} -theories). For a given k -quantifier logic $\mathcal{L}^{\mathcal{Q}}$ and signature σ , we denote the $\mathcal{L}^{\mathcal{Q}}$ -theory of a pointed σ -structure (\mathfrak{A}, α) as

$$\text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{A}, \alpha) := \{\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma) \mid \mathfrak{A}, \alpha \models \varphi\}$$

and the $\mathcal{L}_q^{\mathcal{Q}}$ -theory for a given quantifier rank $q \in \mathbb{N}$ as

$$\text{Th}_{\mathcal{L}_q^{\mathcal{Q}}}^q(\mathfrak{A}, \alpha) := \{\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma) \mid \mathfrak{A}, \alpha \models \varphi, \text{qr}(\varphi) \leq q\}.$$

Similarly, for $s \subseteq A^k$, we define

$$\begin{aligned} \text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{A}, s) &:= \{\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma) \mid \mathfrak{A}, s \models \varphi\}, \\ \text{Th}_{\mathcal{L}_q^{\mathcal{Q}}}^q(\mathfrak{A}, s) &:= \{\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma) \mid \mathfrak{A}, s \models \varphi, \text{qr}(\varphi) \leq q\}. \end{aligned}$$

Example 2.8 (modal quantifiers as 1-quantifiers). For a given binary relation $R \subseteq A \times A$, we denote the set of successors of a given element $a \in A$ as

$$R[a] := \{c \in A \mid (a, c) \in R\}.$$

We start by defining some well-known 1-quantifiers that come up in a modal setting, as summarised in the following table in which R denotes a binary relation.

Q	semantics	σ_Q	reference
\diamond	there is an R -successor s.t.	(R)	[1]
$\diamond^{\geq k}$	there are at least k R -successors s.t.	(R)	[25]
\forall	all elements satisfy	\emptyset	[11]
\exists	there is an element s.t.	\emptyset	[11]
\circ	there is a cycle of length ≥ 3 of elements s.t.	(R)	
\bullet	there are infinitely many reflexive R -successors s.t.	(R)	[24]
\hookrightarrow	there is a reachable element s.t.	(R)	[22]

These are obtained as 1-quantifiers with the following witness sets:

$$\begin{aligned} \diamond(\mathfrak{A}, a) &:= \{\{c\} \mid c \in R^{\mathfrak{A}}[a]\}, \\ \diamond^{\geq k}(\mathfrak{A}, a) &:= \{s \subseteq A \mid |s \cap R^{\mathfrak{A}}[a]| \geq k\}, \\ \forall(\mathfrak{A}, a) &:= \{A\}, \\ \exists(\mathfrak{A}, a) &:= \{s \subseteq A \mid s \neq \emptyset\}, \\ \circ(\mathfrak{A}, a) &:= \{\{c_0, \dots, c_{l-1}\} \subseteq A \mid l \geq 3 \text{ s.t. f.a. } i \in \mathbb{Z}_l : (c_i, c_{i+1}) \in R^{\mathfrak{A}}\}, \\ \bullet(\mathfrak{A}, a) &:= \{s \subseteq R^{\mathfrak{A}}[a] \mid |s| = \infty, \text{ f.a. } c \in s : (c, c) \in R^{\mathfrak{A}}\}, \\ \hookrightarrow(\mathfrak{A}, a) &:= \{\{c_l\} \subseteq A \mid l \geq 0, c_0, \dots, c_l \in A \\ &\quad \text{s.t. } c_0 = a \text{ and f.a. } 0 \leq i < l : (c_i, c_{i+1}) \in R^{\mathfrak{A}}\}. \end{aligned}$$

Example 2.9 (monotonic neighbourhood semantics versus 1-quantifiers). More generally, we can capture monotonic neighbourhood semantics in terms of 1-quantifier logics.

A neighbourhood model $\mathfrak{M} := (W, N, V)$ consists of a universe W , a neighbourhood function $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$, and a valuation $V : p_i \mapsto P_i \in \mathcal{P}(W)$,

defining the relational interpretation for the basic propositions p_i . The standard logic is modelled on basic modal language, with standard semantics for atomic formulae, propositional connectives, and

$$\mathfrak{M}, w \models \Box\varphi \quad :\iff \quad \{u \in W \mid \mathfrak{M}, u \models \varphi\} \in N(w).$$

For a thorough introduction to neighbourhood semantics, we refer the reader to [21]. A neighbourhood model is monotonic if all $N(w)$ are closed under passage to supersets: $N(w) = N(w)^\uparrow$. Let \mathcal{C} be a class of monotonic neighbourhood models such that for every pair $(W, N, V), (W', N', V') \in \mathcal{C}$

$$(W, V) \simeq (W', V') \quad \implies \quad (W, V, N) \simeq (W', V', N'). \quad (1)$$

This allows us to inductively translate any given $\varphi \in \text{ML}$ into the language of 1-quantifier logics akin to the standard translation of basic modal logic over Kripke frames. We translate box as $\text{ST}(\Box\psi) := Q\text{ST}(\psi)$ with witness sets according to $Q((W, V), w) := N(w)$. We find that, for $P_i^{\mathfrak{M}} := V(p_i)$ and $w \in W$,

$$(W, N, V), w \models \varphi \quad \iff \quad (W, (P_i^{\mathfrak{M}})_{i \in \mathbb{N}}), w \models_{\mathcal{L}\{Q\}} \text{ST}(\varphi).$$

In this sense, 1-quantifiers are equivalent to neighbourhood semantics in restriction to a class of models given by the frame condition above.

Note that this construction ‘externalises’ the neighbourhood functions to encode them as 1-quantifiers. Thus, (1) is necessary to avoid the violation of isomorphism invariance of $\mathcal{L}\{Q\}$.

Example 2.10 (first-order and counting quantifiers). Fix some $k \geq 1$ and define

$$Q := \{\exists_{x_i}^{\geq n} \mid n \in \mathbb{N}, 1 \leq i \leq k\}$$

with $\exists_{x_i}^{\geq n}(\mathfrak{A}, \alpha) := \{s \subseteq A^k \mid |s| \geq n, \text{ f.a. } \gamma \in s, j \neq i : \gamma(x_j) = \alpha(x_j)\}$. Then the logic $\mathcal{L}^Q \equiv C^k$ is the k -variable fragment of first-order logic with counting quantifiers; and the restriction to $(\exists_{x_i}^{\geq 1})_{1 \leq i \leq k}$ corresponds to the k -variable fragment FO^k of first-order logic [6, 16, 4, 19].

3 Bisimulation

We aim for natural characterisations of the expressiveness of k -quantifier logics in the traditional model-theoretic tradition of back&forth techniques that capture levels of \mathcal{L}^Q -indistinguishability between structures with assignments. For classical first-order logic FO this is exemplified in the original Ehrenfeucht–Fraïssé approach (cf. [7]), which has many natural restrictions and adaptations for FO^k and C^k , as well as, e.g. the guarded fragment GF of FO, in terms of suitable pebble game equivalences.

In the modal tradition, the same idea arises – more fundamentally, in a way – in the traditional shape of bisimulation games and equivalences. Regarding the concept of back&forth equivalences in the bisimulation format as the most fundamental incarnation of the idea, we use that terminology but, for reasons of historical priority, refer to the crucial links as (adaptations of the) Ehrenfeucht–Fraïssé theorem. For presentational purposes, we also put the natural game intuition, rather than the more extensional formalisation in terms of

back&forth systems, in the foreground. Our adaptation of bisimulation notions for k -quantifier logics is also in line with the natural adaptation of basic modal bisimulation to monotone neighbourhood logics (cf. [21]), or inquisitive modal logic (cf. [3]).

Definition 3.1 ($\mathcal{L}^\mathcal{Q}$ -bisimulation). There are two variants of the $\mathcal{L}^\mathcal{Q}$ -bisimulation game – the unbounded game $G^\mathcal{Q}$ and the q -round game $G_q^\mathcal{Q}$. Both games are played by two players **I** and **II** over two σ -structures \mathfrak{A} and \mathfrak{B} . Positions in the game are pairs of k -pointed σ -structures $(\mathfrak{A}, \alpha; \mathfrak{B}, \beta)$ with $\alpha \in A^k$ and $\beta \in B^k$.

Player **II** loses in any position $(\mathfrak{A}, \alpha; \mathfrak{B}, \beta)$ that violates *atom equivalence*, i.e. in which (\mathfrak{A}, α) and (\mathfrak{B}, β) can be distinguished by a quantifier-free formula.

A round, from position $(\mathfrak{A}, \alpha; \mathfrak{B}, \beta)$, is played as follows:

- **I** picks one of the two pointed structures, say (\mathfrak{A}, α) , a quantifier $Q \in \mathcal{Q}_\sigma$, and a witness set $s \in Q(\mathfrak{A}, \alpha)$.
- **II** must respond with a witness set $t \in Q(\mathfrak{B}, \beta)$.
- **I** must pick some assignment $\delta \in t$.
- **II** must respond with some assignment $\gamma \in s$.
- The new position in the game is $(\mathfrak{A}, \gamma; \mathfrak{B}, \delta)$,

and either player loses during this round when stuck for a required response.

We say that **II** wins a play of $G^\mathcal{Q}$ if **I** loses or if the play continues indefinitely without loss for **II**. Similarly, **II** wins in $G_q^\mathcal{Q}$ if player **I** loses or neither player loses during all q rounds of the game.

We say that the pointed structures (\mathfrak{A}, α) and (\mathfrak{B}, β) are $\mathcal{L}^\mathcal{Q}$ -bisimilar, denoted $(\mathfrak{A}, \alpha) \sim_{\mathcal{L}^\mathcal{Q}} (\mathfrak{B}, \beta)$, if **II** has a winning strategy for $G^\mathcal{Q}$ with initial configuration $(\mathfrak{A}, \alpha; \mathfrak{B}, \beta)$.

Similarly, we say that the k -pointed structures are $\mathcal{L}_q^\mathcal{Q}$ -bisimilar for some $q \in \mathbb{N}$, denoted $(\mathfrak{A}, \alpha) \sim_{\mathcal{L}_q^\mathcal{Q}}^q (\mathfrak{B}, \beta)$, if **II** has a winning strategy for $G_q^\mathcal{Q}$ from this initial position.

As the $\mathcal{L}^\mathcal{Q}$ -bisimulation game is a Borel game, it is determined [17] in the sense that any position of the game admits a winning strategy for precisely one of the two players.

4 An Ehrenfeucht–Fraïssé Theorem

The Ehrenfeucht–Fraïssé theorem for first-order logic FO essentially states a correspondence between indistinguishability by formulae up to a given quantifier-rank q and a winning strategy for **II** in the q -round FO pebble game. The theorem carries over to basic modal logic and, as we will see in this section, also to k -quantifier logics with the corresponding notion of $\mathcal{L}_q^\mathcal{Q}$ -bisimilarity.

Theorem 4.1 (Ehrenfeucht–Fraïssé theorem for $\mathcal{L}^\mathcal{Q}$). *Let σ be a finite signature and $\mathcal{L} := \mathcal{L}^\mathcal{Q}$ a k -quantifier logic with finitely many quantifiers in \mathcal{Q}_σ . For any $q \in \mathbb{N}$ and pair of pointed σ -structures (\mathfrak{A}, α) , (\mathfrak{B}, β) , the following are equivalent:*

- (i) $(\mathfrak{A}, \alpha) \sim_{\mathcal{L}}^q (\mathfrak{B}, \beta)$,

(ii) $(\mathfrak{A}, \alpha) \equiv_{\mathcal{L}}^q (\mathfrak{B}, \beta)$.

The proof of the theorem is a simple adaptation of the classical Ehrenfeucht–Fraïssé argument.

Proof by induction on q . The base case for $q = 0$ is clear. It remains to show the induction step. Let $q > 0$.

We show the implication “(i) \implies (ii)” by syntactic induction. As the claim is preserved under boolean connectives, we just need to consider $\varphi = Q\psi$ with $\text{qr}(\varphi) = q$. Supposing that, e.g. $\mathfrak{A}, \alpha \models \varphi$, we need to show that also $\mathfrak{B}, \beta \models \varphi$. Let $s \in Q(\alpha)$ be a witness for $\mathfrak{A}, \alpha \models Q\psi$ and let **I** pick s as challenge in the $\mathcal{L}^{\mathcal{Q}}$ -bisimulation game.

$\sim_{\mathcal{L}}^q$ guarantees that **II** has a response $t \in Q(\beta)$ in the bisimulation game such that every $\delta \in t$ is $\mathcal{L}_{q-1}^{\mathcal{Q}}$ -bisimilar to some $\gamma \in s$. By the induction hypothesis, $\mathfrak{B}, \delta \models \psi$ iff $\mathfrak{A}, \gamma \models \psi$ as $\text{qr}(\psi) = q - 1$. As s witnesses $Q\psi$ also $\mathfrak{B}, \delta \models \psi$ for all $\delta \in t$, ensuring that t is a witness for $\mathfrak{B}, \beta \models Q\psi$.

For the converse implication “(ii) \implies (i)”, suppose $\text{Th}_{\mathcal{L}^{\mathcal{Q}}}^q(\mathfrak{A}, \alpha) = \text{Th}_{\mathcal{L}^{\mathcal{Q}}}^q(\mathfrak{B}, \beta)$. W.l.o.g. assume **I** picks some $s \in Q(\alpha)$. We need to find a suitable response for **II**.

Since there are only finitely many relations, variables and quantifiers, the set of formulae up to quantifier rank q is finite up to logical equivalence. In particular, the conjunctions and disjunction in

$$\chi := Q \bigvee \bigwedge_{\gamma \in s} \text{Th}_{\mathcal{L}^{\mathcal{Q}}}^{q-1}(\mathfrak{A}, \gamma)$$

are effectively finite. By construction, $\mathfrak{A}, \alpha \models \chi$ is witnessed by s . Since $\text{Th}_{\mathcal{L}^{\mathcal{Q}}}^q(\mathfrak{A}, \alpha) = \text{Th}_{\mathcal{L}^{\mathcal{Q}}}^q(\mathfrak{B}, \beta)$, also $\mathfrak{B}, \beta \models \chi$. Take a witness $t \in Q_{\mathfrak{B}}(\beta)$ for χ . Let t be **II**’s response in the game and $\delta \in t$ be **I**’s challenge. By choice of the witness t , $\mathfrak{B}, \delta \models \bigvee_{\gamma \in s} \bigwedge \text{Th}_{\mathcal{L}^{\mathcal{Q}}}^{q-1}(\mathfrak{A}, \gamma)$. Thus, there exists some $\gamma \in s$ with $\mathfrak{B}, \delta \models \text{Th}_{\mathcal{L}^{\mathcal{Q}}}^{q-1}(\mathfrak{A}, \gamma)$ for **II** to play. \square

Note that the implication “(i) \implies (ii)” even holds for infinitely many relations and quantifiers, while the finiteness conditions are crucial for the converse direction due to the following observation that is easily shown by induction on q .

Observation 4.2. *Let $\mathcal{L} := \mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic with finite \mathcal{Q} and σ a finite signature. Then the equivalence relations $\equiv_{\mathcal{L}}^q, \sim_{\mathcal{L}}^q$ are of finite index.*

Corollary 4.3 ($\mathcal{L}^{\mathcal{Q}}$ -bisimulation invariance). *For a k -quantifier logic $\mathcal{L}^{\mathcal{Q}}$ and signature σ , let (\mathfrak{A}, α) and (\mathfrak{B}, β) be $\mathcal{L}^{\mathcal{Q}}$ -bisimilar k -pointed σ -structures. Then*

$$\mathfrak{A}, \alpha \equiv_{\mathcal{L}^{\mathcal{Q}}} \mathfrak{B}, \beta, \text{ or, equivalently, } \text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{A}, \alpha) = \text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{B}, \beta).$$

Proof. This follows immediately from Theorem 4.1 as $\mathcal{L}^{\mathcal{Q}}$ -bisimilarity implies $\mathcal{L}_q^{\mathcal{Q}}$ -bisimilarity for all q , and as each individual $\mathcal{L}^{\mathcal{Q}}$ -formula refers to just finitely many relation symbols and quantifiers, and is of finite quantifier rank. \square

The *characteristic formulae* $\chi_{\mathfrak{A},\alpha}^q$, for k -pointed σ -structures (\mathfrak{A}, α) and quantifier-rank q , are built up inductively so as to capture the challenge-response requirements that enable player **II** to stay abreast of player **I** for so many further rounds. These characteristic formulae are such that

$$\mathfrak{B}, \beta \models \chi_{\mathfrak{A},\alpha}^q \iff (\mathfrak{A}, \alpha) \sim_{\mathcal{L}^Q}^q (\mathfrak{B}, \beta). \quad (2)$$

The quantifier-rank 0 formula $\chi_{\mathfrak{A},\alpha}^0$ just specifies the atomic type of the k -assignment α in \mathfrak{A} (by a finite conjunction of relational atoms and negated atoms), as a guarantee of atom equivalence on the righthand side of (2).

For $q > 0$, a characteristic formula is the conjunction of

- the quantifier-rank 0 formula $\chi_{\mathfrak{A},\alpha}^0$ in order to ensure atom equivalence in (2),
- conditions ensuring the existence of *forth responses* for **II** to all potential challenges by **I** played in \mathfrak{A} , and
- conditions ensuring the existence of *back responses* for **II** to all potential challenges by **I** played in \mathfrak{B} in the context of (2).

Thus, the characteristic formula for quantifier-rank $q + 1$ is

$$\begin{aligned} \chi_{\mathfrak{A},\alpha}^{q+1} := & \chi_{\mathfrak{A},\alpha}^0 \wedge \bigwedge_{Q \in \mathcal{Q}_\sigma} \underbrace{\left(\bigwedge_{s \in Q(\alpha)} Q \bigvee_{\gamma \in s} \chi_{\mathfrak{A},\gamma}^q \right)}_{\text{“forth responses”}} \\ & \wedge \underbrace{\bigwedge \left\{ \neg Q \bigvee \Phi \mid \Phi \subseteq \Delta_q, \mathfrak{A}, \alpha \models \neg Q \bigvee \Phi \right\}}_{\text{“back responses”}} \end{aligned}$$

where $\Delta_q := \{\chi_{\mathfrak{B},\beta}^q \mid (\mathfrak{B}, \beta) \text{ } k\text{-pointed } \sigma\text{-structure}\}$ is the set of all characteristic formulae at level q .

Proposition 4.4 (characteristic formulae). *For a finite signature σ , k -quantifier logic \mathcal{L}^Q with finite \mathcal{Q}_σ , and k -pointed σ -structures (\mathfrak{A}, α) and (\mathfrak{B}, β) ,*

$$\mathfrak{B}, \beta \models \chi_{\mathfrak{A},\alpha}^q \iff (\mathfrak{B}, \beta) \sim_{\mathcal{L}^Q}^q (\mathfrak{A}, \alpha).$$

Proof. We only show the forward implication as the converse direction is obvious by Theorem 4.1 since $\mathfrak{A}, \alpha \models \chi_{\mathfrak{A},\alpha}^q$. Our strategy is to show by induction on q that $\mathfrak{B}, \beta \models \chi_{\mathfrak{A},\alpha}^q$ implies that **II** has a winning strategy for q rounds in the game from $(\mathfrak{A}, \alpha; \mathfrak{B}, \beta)$. This is obvious for $q = 0$ (the base case).

Assuming $\mathfrak{B}, \beta \models \chi_{\mathfrak{A},\alpha}^{q+1}$, first consider a first round in which **I** first chooses some $s \in Q(\mathfrak{A}, \alpha)$. The *forth*-part in $\chi_{\mathfrak{A},\alpha}^{q+1}$ ensures that $\mathfrak{B}, \beta \models Q \bigvee_{\gamma \in s} \chi_{\mathfrak{A},\gamma}^q$. Thus, we can let **II** pick a witness $t \in Q(\mathfrak{B}, \beta)$ such that $\mathfrak{B}, t \models \bigvee_{\gamma \in s} \chi_{\mathfrak{A},\gamma}^q$, which clearly allows **II** a suitable response to any challenge $d \in t$ that **I** may select.

If **I** first chooses some $t \in Q(\mathfrak{B}, \beta)$, on the other hand, we look to the *back*-part in $\chi_{\mathfrak{A},\alpha}^{q+1}$: let $\Phi := \{\chi_{\mathfrak{B},\delta}^q \mid \delta \in t\}$ so that $\mathfrak{B}, \beta \models Q \bigvee \Phi$; it follows that $\mathfrak{A}, \alpha \models Q \bigvee \Phi$, too, since otherwise $\neg Q \bigvee \Phi$ would be a conjunct of $\chi_{\mathfrak{A},\alpha}^{q+1}$. If **II** chooses a witness $s \in Q(\mathfrak{A}, \alpha)$ for $\mathfrak{A}, \alpha \models Q \bigvee \Phi$, then **II** has safe responses to challenges by **I** in the second phase of this round. \square

The existence of characteristic formulae directly implies a normal form according to

$$\varphi \equiv \bigvee \{ \chi_{\mathfrak{A}, \alpha}^{\text{qr}(\varphi)} \mid \mathfrak{A}, \alpha \models \varphi \}.$$

Note that the disjunction on the right is finite up to logical equivalence due to Observation 4.2.

5 A Hennessy–Milner Theorem

We want to find sufficient conditions for the converse of the implication in Corollary 4.3. The classical Hennessy–Milner theorem for basic modal logic ML states that ML-equivalence between pointed Kripke structures implies their bisimilarity if the underlying frames are finitely branching; this can be substantially generalised to the class of all ML-saturated structures (cf. [10]). Furthermore, the proof has been adapted to neighbourhood semantics in [12, Chapter 4].

We here adapt this result to k -quantifier logics by showing that $\mathcal{L}^{\mathcal{Q}}$ -equivalent k -pointed σ -structures that are $\mathcal{L}^{\mathcal{Q}}$ -saturated (as in the following definition) are $\mathcal{L}^{\mathcal{Q}}$ -bisimilar.

Definition 5.1 (Q -type). Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic and \mathfrak{A} a σ -structure. For every $Q \in \mathcal{Q}$ and $\alpha \in A^k$, we call $\Psi \subseteq \mathcal{L}^{\mathcal{Q}}(\sigma)$ a (partial) Q -type of (\mathfrak{A}, α) if $\mathfrak{A}, \alpha \models Q \wedge \Psi_0$ for all finite $\Psi_0 \subseteq \Psi$. We say that Ψ is *realised* by $s \in Q(\alpha)$ if $\mathfrak{A}, s \models \Psi$ (with flat semantics as introduced in Definition 2.3).

Similarly, $\Psi \subseteq \mathcal{L}^{\mathcal{Q}}(\sigma)$ is an s -type of a given $s \in Q(\alpha)$ if for all finite $\Psi_0 \subseteq \Psi$ there is a $\gamma \in s$ with $\mathfrak{A}, \gamma \models \Psi_0$. We say that Ψ is *realised* by $\gamma \in s$ if $\mathfrak{A}, \gamma \models \Psi$.

We call \mathfrak{A} $\mathcal{L}^{\mathcal{Q}}$ -saturated if for all $Q \in \mathcal{Q}$ and $\alpha \in A^k$ every Q -type of (\mathfrak{A}, α) and every s -type of every $s \in Q(\alpha)$ is realised.

We postpone the construction of such saturated structures to Section 7, where we find saturated extensions for k -quantifier logics that are compatible with ultrapowers (Lemma 7.2).

To better understand the importance of saturation, consider the standard example of the two pointed Kripke structures in Figure 1. They have the same theory w.r.t. basic modal logic $\text{ML} \equiv \mathcal{L}^{\{\diamond\}}$ as **II** can win any finite $\mathcal{L}^{\{\diamond\}}$ -bisimulation game; but they are not $\mathcal{L}^{\{\diamond\}}$ -bisimilar, as **I** may choose the witness $\{b^*\}$ and is thus able to continue picking successors indefinitely, while **II** gets stuck after $n + 1$ rounds if $\{a_n\}$ was selected as the first response.

The issue appears to be that \mathfrak{B} possesses a richer internal structure than \mathfrak{A} , and indeed, \mathfrak{B} is $\mathcal{L}^{\{\diamond\}}$ -saturated, while \mathfrak{A} is not. Such imbalance cannot occur between two $\mathcal{L}^{\{\diamond\}}$ -saturated structures, and this phenomenon naturally also adapts to the scenario of k -quantifier logics as shown in Theorem 5.2 below.

Note that the modal case is special as it just requires saturation w.r.t. \diamond -types since s -types are always realised due to them being essentially (supersets of) singleton sets. Hennessy–Milner results for neighbourhood logic and for inquisitive modal logic, which are closer in spirit to the present more general setting, are treated in [12] and [18], respectively.

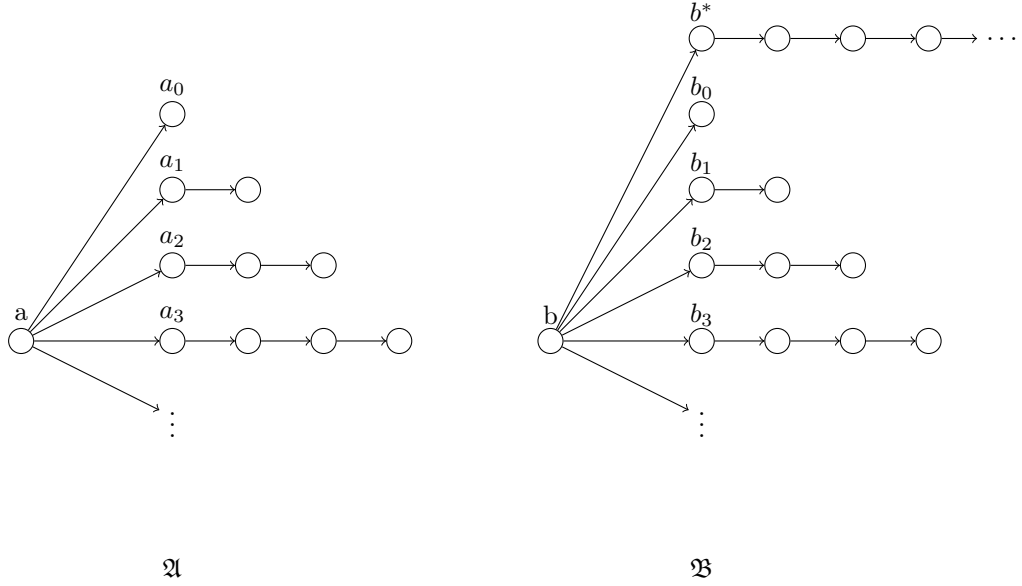


Figure 1: The structures (\mathfrak{A}, a) and (\mathfrak{B}, b) have the same ML-theory, but they are not ML-bisimilar.

Theorem 5.2 (a Hennessy–Milner Theorem). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic and (\mathfrak{A}, α) , (\mathfrak{B}, β) a pair of pointed σ -structures. If (\mathfrak{A}, α) and (\mathfrak{B}, β) are $\mathcal{L}^{\mathcal{Q}}$ -saturated then*

$$(\mathfrak{A}, \alpha) \equiv_{\mathcal{L}^{\mathcal{Q}}} (\mathfrak{B}, \beta) \implies (\mathfrak{A}, \alpha) \sim_{\mathcal{L}^{\mathcal{Q}}} (\mathfrak{B}, \beta).$$

Proof. We show that player **II** can maintain $\mathcal{L}^{\mathcal{Q}}$ -equivalence, in response to any challenges by **I**, as an invariant that guarantees a win in any infinite play. Let w.l.o.g. $s \in Q(\alpha)$ be **I**'s challenge in the $\mathcal{L}^{\mathcal{Q}}$ -bisimulation game. Consider

$$\Psi := \left\{ \bigvee_{\gamma \in s} \bigwedge \chi_{\mathfrak{A}, \gamma}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0)) \mid q \in \mathbb{N}, \sigma_0 \subseteq \sigma \text{ finite}, \mathcal{Q}_0 \subseteq \mathcal{Q} \text{ finite} \right\}$$

where $\chi_{\mathfrak{A}, \gamma}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0))$ is the characteristic formula for (\mathfrak{A}, γ) at quantifier rank q with respect to the fragment $\mathcal{L}^{\mathcal{Q}_0}(\sigma_0)$. Note that the conjunctions and disjunctions in Ψ are finite up to logical equivalence as we restrict the language to finite signatures and sets of quantifiers. The set Ψ is by definition a Q -type of (\mathfrak{A}, α) and thus of (\mathfrak{B}, β) as $(\mathfrak{A}, \alpha) \equiv_{\mathcal{L}} (\mathfrak{B}, \beta)$. Since \mathfrak{B} is saturated, there is a $t \in Q_{\mathfrak{B}}(\beta)$ realising Ψ , i.e. $\mathfrak{B}, \delta \models \Psi$ for all $\delta \in t$.

Let t be **II**'s response and let $\delta \in t$ be **I**'s challenge for the second phase of this round. By choice of t , for every $q \in \mathbb{N}$ and finite $\sigma_0 \subseteq \sigma$, $\mathcal{Q}_0 \subseteq \mathcal{Q}$ there is some $\gamma \in s$ such that $\chi_{\mathfrak{B}, \delta}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0)) = \chi_{\mathfrak{A}, \gamma}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0))$. As every finite subset $\Psi_0 \subseteq \text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{B}, \delta)$ only contains finitely many symbols and quantifiers, it is, by Theorem 4.1 and Proposition 4.4, a consequence of some $\chi_{\mathfrak{B}, \delta}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0))$. Therefore, the theory $\text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{B}, \delta) \equiv \bigcup_{q, \sigma_0, \mathcal{Q}_0} \chi_{\mathfrak{B}, \delta}^q(\mathcal{L}^{\mathcal{Q}_0}(\sigma_0))$ is an s -type of \mathfrak{A} . As \mathfrak{A} is saturated, there must be some $\gamma \in s$ with $\text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{B}, \delta) = \text{Th}_{\mathcal{L}^{\mathcal{Q}}}(\mathfrak{A}, \gamma)$, which it is safe for **II** to play. \square

6 The Łoś Property and First-Order Logic

We quickly recall some definitions related to ultraproducts. For a more thorough introduction, we refer to [13, Chapter 9.5].

A collection $\mathcal{F} \subseteq \mathcal{P}(I) \setminus \{\emptyset\}$ of nonempty subsets is a *filter* over the index set I if it is closed under passage to supersets (upward closed: $\mathcal{F}^\uparrow = \mathcal{F}$) and under finite intersections ($J_1, J_2 \in \mathcal{F} \Rightarrow J_1 \cap J_2 \in \mathcal{F}$, or $\mathcal{F} \cap \mathcal{F} = \mathcal{F}$). An *ultrafilter* over I is a filter \mathcal{U} such that, for every $J \in \mathcal{P}(I)$, either $J \in \mathcal{U}$ or $I \setminus J \in \mathcal{U}$. The latter is a maximality condition w.r.t. the \subseteq -relation (over $\mathcal{P}(\mathcal{P}(I))$) among filters, and an application of Zorn's lemma shows that every filter \mathcal{F} can be extended to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$. An ultrafilter extending the filter $\{\{i\}\}$ for some $i \in I$ is called *principal* and *non-principal* otherwise. For $I = \mathbb{N}$, every non-principal ultrafilter contains the Fréchet filter, consisting of all co-finite subsets.

The direct product of a family $(\mathfrak{A}_i)_{i \in I}$ of σ -structures is denoted as $\prod_{i \in I} \mathfrak{A}_i$. We use boldface letters like \mathbf{a} for its elements $\mathbf{a} = (a_i)_{i \in I} \in \prod_{i \in I} A_i$, and extend this notational convention to tuples of elements $\mathbf{a} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(n)}) \in (\prod_{i \in I} A_i)^n$, where we then also write $\mathbf{a}_i := (a_i^{(1)}, \dots, a_i^{(n)}) \in A_i^n$ for the tuple of entries in the i th component of the product. Direct products over constant families, $\mathfrak{A}_i = \mathfrak{A}$ for all $i \in I$, are denoted \mathfrak{A}^I and addressed as direct powers. The \mathcal{F} -*reduced product* of the family (\mathfrak{A}_i) w.r.t. the filter \mathcal{F} is obtained as a σ -structure $\prod \mathfrak{A}_i / \mathcal{F}$ over the quotient set $\prod_{i \in I} A_i / \sim_{\mathcal{F}}$. The equivalence relation

$$\mathbf{a} \sim_{\mathcal{F}} \mathbf{a}' \text{ if } \{i \in I \mid a_i = a'_i\} \in \mathcal{F}$$

identifies two elements $\mathbf{a} = (a_i)_{i \in I}$ and $\mathbf{a}' = (a'_i)_{i \in I}$ of the direct product precisely if they agree in \mathcal{F} -many components. We denote $\sim_{\mathcal{F}}$ -equivalence classes of elements \mathbf{a} as $\mathbf{a}_{\mathcal{F}} \in \prod A_i / \sim_{\mathcal{F}}$, and again extend this notation to n -tuples. Note that for $\mathbf{a}^{(1)}, \mathbf{b}^{(1)}, \dots, \mathbf{a}^{(k)}, \mathbf{b}^{(k)} \in \prod A_i$,

$$(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)}) \sim_{\mathcal{F}} (\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(k)}) \iff \mathbf{a}^{(1)} \sim_{\mathcal{F}} \mathbf{b}^{(1)}, \dots, \mathbf{a}^{(k)} \sim_{\mathcal{F}} \mathbf{b}^{(k)}.$$

The interpretation of the relation symbols $R \in \sigma$ in $\mathfrak{C} := \prod \mathfrak{A}_i / \sim_{\mathcal{F}}$ is defined according to

$$\mathbf{a} \in R^{\mathfrak{C}} \text{ if } \{i \in I \mid \mathbf{a}_i \in R^{\mathfrak{A}_i}\} \in \mathcal{F},$$

representing membership in \mathcal{F} -many components (which is well-defined, i.e. independent of representatives, in the quotient, due to closure of \mathcal{F} under finite intersections). Notation like $\mathfrak{A}^I / \mathcal{F}$ or $\mathfrak{A}^I / \mathcal{U}$ correspondingly refers to *reduced powers* or *ultrapowers*, i.e. reduced products w.r.t. a filter or ultrafilter for a family of structures with constant entry \mathfrak{A} for the component structures \mathfrak{A}_i . All of these conventions extend naturally to σ -structures with assignments $\alpha_i := \mathbf{a} \in A_i^n$ as component objects. Already for the definition of the quotient and the interpretation of the relation symbols it is suggestive to evaluate formulae over the direct product $\prod_{i \in I} (\mathfrak{A}_i, \alpha_i)$, with assignments $\alpha = \mathbf{a} \in (\prod_{i \in I} A_i)^n$, in the boolean powerset algebra $\mathcal{P}(I)$, giving

$$\llbracket \varphi \rrbracket_{\alpha} := \{i \in I \mid \mathfrak{A}_i, \alpha_i \models \varphi\} \in \mathcal{P}(I)$$

as a “truth value” to φ over $\prod_{i \in I} (\mathfrak{A}_i, \alpha_i)$. Then the interpretation of the relations in the reduced product $\prod \mathfrak{A}_i / \mathcal{F}$ amounts to the following equivalence for all

atomic formulae φ :

$$\prod \mathfrak{A}_i/\mathcal{F}, \alpha_{\mathcal{F}} \models \varphi \iff \llbracket \varphi \rrbracket \in \mathcal{F}.$$

The classical theorem of Łoś states precisely this equivalence for all $\varphi \in \text{FO}$ in the case of ultraproducts.

Definition 6.1 (Łoś property). We say that an abstract logic \mathcal{L} has the *Łoś property* if for every pointed ultraproduct $(\prod \mathfrak{A}_i/\mathcal{U}, \alpha_{\mathcal{U}})$ and formula $\varphi \in \mathcal{L}(\sigma)$,

$$\prod \mathfrak{A}_i/\mathcal{U}, \alpha_{\mathcal{U}} \models \varphi \iff \llbracket \varphi \rrbracket_{\alpha} \in \mathcal{U}.$$

Theorem 6.2 (Łoś [13, Theorem 9.5.1]). *First-order logic has the Łoś property.*

Definition 6.3 (compactness property). We say that an abstract logic \mathcal{L} has the *compactness property* if every *finitely satisfiable* set of formulae is *satisfiable*, i.e. for every set of formulae $\Phi \subseteq \mathcal{L}(\sigma)$ the following are equivalent:

- (i) Φ has a model $\mathfrak{A}, \alpha \models \Phi$, and
- (ii) every finite $\Phi_0 \subseteq \Phi$ has a model $\mathfrak{A}_0, \alpha_0 \models \Phi_0$.

It is a well-known fact that for abstract logics the Łoś property implies compactness (see, e.g., [2, Corollary 4.1.11] for the FO version).

There is a set-theoretically much deeper result due to Keisler and Shelah stating that two structures are indistinguishable by first-order formulae if, and only if, they possess isomorphic ultrapowers [2, Theorem 6.1.15].

As a corollary to the theorem, having the Łoś property is equivalent to being a fragment of first-order logic. Put differently, in the context of Definition 2.1, first-order logic is the global maximum among all abstract logics having the Łoś property.

Lemma 6.4. *Let \mathcal{L}_1 and \mathcal{L}_2 be two abstract logics that have the Łoś property and \mathcal{L} the FO-closure of $\mathcal{L}_1 \cup \mathcal{L}_2$, i.e. $\mathcal{L}(\sigma)$ is the closure of $\mathcal{L}_1(\sigma) \cup \mathcal{L}_2(\sigma)$ under boolean connectives, universal and existential quantification. Then \mathcal{L} also has the Łoś property.*

Proof by syntactic induction. The proof is very similar to the usual proof of the Łoś property for first-order logic. The base case, where $\varphi \in \mathcal{L}_1 \cup \mathcal{L}_2$, is given by assumption. For the induction step, we use the maximality of \mathcal{U} to show compatibility with negation and disjunction, while closure under finite intersections and supersets takes care of conjunctions. The step for the existential quantifier also works as usual. \square

Lemma 6.5. *Let $\hat{\mathcal{L}}$ be an abstract logic extending the abstract logic \mathcal{L} . Furthermore, let $\varphi \in \mathcal{L}(\sigma)$, $\psi \in \hat{\mathcal{L}}(\sigma)$ and $T \subseteq \mathcal{L}(\sigma)$. If for all $\chi \in \mathcal{L}(\sigma)$, $T \not\models \psi \leftrightarrow \chi$ then*

$$\begin{aligned} & T \cup \{\varphi\} \not\models \psi \leftrightarrow \chi \quad \text{for all } \chi \in \mathcal{L}(\sigma) \\ \text{or} \quad & T \cup \{\neg\varphi\} \not\models \psi \leftrightarrow \chi \quad \text{for all } \chi \in \mathcal{L}(\sigma). \end{aligned}$$

Proof. Towards a contradiction, assume there exist formulae χ and χ' such that $T \models \varphi \rightarrow (\psi \leftrightarrow \chi)$ and $T \models \neg\varphi \rightarrow (\psi \leftrightarrow \chi')$. Then $T \models \eta$ for η given as $\eta := \psi \leftrightarrow ((\varphi \wedge \chi) \vee (\neg\varphi \wedge \chi'))$, contradicting our assumption as η is in $\mathcal{L}(\sigma)$. \square

Lemma 6.6. *Let $\hat{\mathcal{L}}$ be a compact abstract logic extending an abstract logic \mathcal{L} and $\varphi \in \hat{\mathcal{L}} \setminus \mathcal{L}$ not expressible in \mathcal{L} . Then there are pointed σ -structures (\mathfrak{A}, α) and (\mathfrak{B}, β) with*

$$\text{Th}_{\mathcal{L}}(\mathfrak{A}, \alpha) = \text{Th}_{\mathcal{L}}(\mathfrak{B}, \beta),$$

while $\mathfrak{A}, \alpha \models_{\hat{\mathcal{L}}} \varphi$ and $\mathfrak{B}, \beta \models_{\hat{\mathcal{L}}} \neg\varphi$.

Proof. Compactness of $\hat{\mathcal{L}}$ ensures that, in the situation of Lemma 6.5, the sets T are closed under unions of \subseteq -chains. Applying Zorn's Lemma yields a \subseteq -maximal such T . Lemma 6.5 ensures that T is complete. Furthermore, $T \cup \{\varphi\}$ as well as $T \cup \{\neg\varphi\}$ are satisfiable, as φ would otherwise be equivalent to \top or \perp under T . \square

Combining these observations, we obtain the following corollary to the theorem by Keisler and Shelah.

Corollary 6.7. *Let \mathcal{L} be an abstract logic. Then the following are equivalent:*

- (i) \mathcal{L} has the Łoś property.
- (ii) $\mathcal{L} \preceq \text{FO}$.

Proof. That (ii) implies (i) is obvious. For (i) \Rightarrow (ii) we give an indirect argument. Towards a contradiction, assume that \mathcal{L} has the Łoś property, but there is some $\varphi \in \mathcal{L}(\sigma)$ that is not expressible in FO. Let $\hat{\mathcal{L}}$ be the FO-closure of $\mathcal{L} \cup \text{FO}$. Then, by Lemma 6.4, $\hat{\mathcal{L}}$ also has the Łoś property and is thus compact. Therefore, we can apply Lemma 6.6 to find pointed σ -structures (\mathfrak{A}, α) and (\mathfrak{B}, β) with $\text{Th}_{\text{FO}}(\mathfrak{A}, \alpha) = \text{Th}_{\text{FO}}(\mathfrak{B}, \beta)$, while $\mathfrak{A}, \alpha \models \varphi$ and $\mathfrak{B}, \beta \models \neg\varphi$.

As (\mathfrak{A}, α) and (\mathfrak{B}, β) are indistinguishable for FO, we may apply Keisler–Shelah to obtain two isomorphic ultrapowers $(\prod \mathfrak{A}/\mathcal{U}, \alpha_{\mathcal{U}}) \cong (\prod \mathfrak{B}/\mathcal{U}, \beta_{\mathcal{U}})$ over \mathfrak{A} and \mathfrak{B} , respectively. The Łoś property of \mathcal{L} ensures that $\prod \mathfrak{A}/\mathcal{U}, \alpha_{\mathcal{U}} \models \varphi$, while $\prod \mathfrak{B}/\mathcal{U}, \beta_{\mathcal{U}} \models \neg\varphi$, contradicting the isomorphism invariance of \mathcal{L} . Thus, \mathcal{L} has to be a fragment of FO. \square

For k -quantifier logics, we obtain an even stronger result in the sense that there is a uniform translation of every quantifier into first-order logic if the corresponding k -quantifier logic has the Łoś property.

Theorem 6.8 (uniform FO-translation). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic that has the Łoś property. Then for every quantifier $Q \in \mathcal{Q}$ and k -ary relation symbol $R \notin \sigma_Q$ there is a formula $\eta \in \text{FO}(\sigma_Q \cup \{R\})$ with k free variables such that for all signatures $\sigma \supseteq \sigma_Q$ with $R \notin \sigma$, pointed σ -structures (\mathfrak{A}, α) and $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$*

$$\mathfrak{A}, \alpha \models_{\mathcal{L}^{\mathcal{Q}}} Q\varphi \iff \mathfrak{A}, \alpha \models_{\text{FO}} \eta[\varphi^*/R]$$

where $\varphi^* \in \text{FO}(\sigma)$ is a translation of φ to first-order logic.

In particular, this yields a uniform translation from $\mathcal{L}^{\mathcal{Q}}$ into FO.

Proof. By Corollary 6.7, we know that $\mathcal{L}^{\mathcal{Q}}$ is a fragment of FO. Thus, let $\eta \in \text{FO}(\sigma_Q \cup \{R\})$ be a translation of $QR(x_1, \dots, x_k) \in \mathcal{L}^{\mathcal{Q}}(\sigma_Q \cup \{R\})$ into FO.

Take any $\sigma \supseteq \sigma_Q$, a fresh relation symbol $R' \notin \sigma$, $\varphi \in \mathcal{L}^{\mathcal{Q}}(\sigma)$ with translation $\varphi^* \in \text{FO}(\sigma)$ and a pointed σ -structure (\mathfrak{A}, α) . Define

$$R'^{\mathfrak{A}} := R^{\mathfrak{A} \upharpoonright \sigma_Q} := \{\gamma \in A^k \mid \mathfrak{A}, \gamma \models \varphi\}.$$

Then, as k -quantifier logics are well-behaved w.r.t. expansions,

$$\begin{aligned} \mathfrak{A}, \alpha \models_{\mathcal{L}^{\mathcal{Q}}} Q\varphi &\iff \mathfrak{A}, \alpha, R'^{\mathfrak{A}} \models_{\mathcal{L}^{\mathcal{Q}}} QR'(x_1, \dots, x_k) \\ &\iff \mathfrak{A} \upharpoonright \sigma_Q, \alpha, R^{\mathfrak{A} \upharpoonright \sigma_Q} \models_{\mathcal{L}^{\mathcal{Q}}} QR(x_1, \dots, x_k) \\ &\iff \mathfrak{A} \upharpoonright \sigma_Q, \alpha, R^{\mathfrak{A} \upharpoonright \sigma_Q} \models_{\text{FO}} \eta. \end{aligned}$$

First-order logic is invariant under renaming of relations and well-behaved w.r.t. expansions. Thus, we obtain

$$\begin{aligned} \mathfrak{A}, \alpha \models_{\mathcal{L}^{\mathcal{Q}}} Q\varphi &\iff \mathfrak{A} \upharpoonright \sigma_Q, \alpha, R'^{\mathfrak{A}} \models_{\text{FO}} \eta[R'/R] \\ &\iff \mathfrak{A}, \alpha \models_{\text{FO}} \eta[\varphi^*/R]. \quad \square \end{aligned}$$

7 A Lindström theorem

In this section, we establish the maximality of k -quantifier logics that have the Łoś property under the corresponding notion of bisimulation invariance and the Łoś property itself. Crucially, we do *not* refer to Keisler–Shelah for the proof, but make do with much more elementary arguments.

Lemma 7.1 (lifting witnesses). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic that has the Łoś property, $Q \in \mathcal{Q}$, and $(\mathfrak{A}, \alpha_{\mathcal{U}}) := (\prod_{i \in I} \mathfrak{A}_i / \mathcal{U}, \alpha_{\mathcal{U}})$ a k -pointed ultraproduct based on $(\mathfrak{A}_i)_{i \in I}$ and ultrafilter \mathcal{U} over I . Then for any $J \in \mathcal{U}$ and family of witnesses $(s_i)_{i \in J}$ with $s_i \in Q(\mathfrak{A}_i, \alpha_i)$, there is an $s \in Q(\mathfrak{A}, \alpha_{\mathcal{U}})$ and $K \in \mathcal{U}$ such that*

$$s \subseteq \{\gamma_{\mathcal{U}} \mid \gamma_i \in s_i \text{ for all } i \in J \cap K\}.$$

Proof. Consider for all $i \in I$ expansions \mathfrak{A}_i^* of \mathfrak{A}_i by a fresh relation $R \notin \sigma$ of arity k via

$$R^{\mathfrak{A}_i^*} := \begin{cases} s_i & \text{if } i \in J, \\ \emptyset & \text{otherwise.} \end{cases}$$

The ultraproduct $\mathfrak{A}^* := \prod_{i \in I} \mathfrak{A}_i^* / \mathcal{U}$ is an expansion of \mathfrak{A} by R and, by definition of k -quantifiers (Definition 2.2), the witness sets $Q(\mathfrak{A}, \alpha_{\mathcal{U}})$ and $Q(\mathfrak{A}^*, \alpha_{\mathcal{U}})$ are identical. Thus, it suffices to find a suitable $s \in Q(\mathfrak{A}^* / \mathcal{U}, \alpha_{\mathcal{U}})$.

By construction, $\mathfrak{A}_i^*, \alpha_i \models QR(x_1, \dots, x_k)$ is witnessed by $s_i \in Q(\mathfrak{A}_i, \alpha_i)$ for all $i \in J$. The Łoś property of $\mathcal{L}^{\mathcal{Q}}$ thus guarantees that $\mathfrak{A}^*, \alpha_{\mathcal{U}} \models QR(x_1, \dots, x_k)$, witnessed by some witness $s \in Q(\mathfrak{A}, \alpha_{\mathcal{U}}) = Q(\mathfrak{A}^*, \alpha_{\mathcal{U}})$ with $s \subseteq R^{\mathfrak{A}^*}$. Using the Łoś property, we find a $K \in \mathcal{U}$ s.t.

$$s \subseteq R^{\mathfrak{A}^*} = \{\gamma_{\mathcal{U}} \mid \gamma_i \in s_i \text{ for all } i \in J \cap K\}. \quad \square$$

By combining our previous insights, we are now in a position to show the existence of $\mathcal{L}^{\mathcal{Q}}$ -saturated structures by rather pedestrian arguments in the case of countable σ and \mathcal{Q} .

Lemma 7.2 (saturated extensions). *Let $\mathcal{L} := \mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic with countably many k -quantifiers that has the Łoś property and \mathfrak{A} a σ -structure for countable σ . Then for every non-principal ultrafilter \mathcal{U} over \mathbb{N} , $\mathfrak{A}^* := \mathfrak{A}^{\mathbb{N}}/\mathcal{U}$ is an \mathcal{L} -saturated \mathcal{L} -elementary extension of \mathfrak{A} .*

Proof. Using the Łoś property, it is easy to verify that \mathfrak{A}^* is an \mathcal{L} -elementary extension via the embedding $\iota : a \mapsto [(a)_{i \in I}]_{\mathcal{U}}$.

Let Ψ be any Q -type of $(\mathfrak{A}^*, \alpha_{\mathcal{U}})$. As $\mathcal{L}(\sigma)$ and thus $\Psi \subseteq \mathcal{L}(\sigma)$ is countable, we may enumerate it as $(\delta_j)_{j \in \mathbb{N}}$ and consider for every $k \in \mathbb{N}$ the set $J_k := \llbracket Q(\delta_0 \wedge \dots \wedge \delta_k) \rrbracket \cap \{i \in I \mid i \geq k\}$. All the J_k are in \mathcal{U} as Ψ is a Q -type of $(\mathfrak{A}, \alpha_{\mathcal{U}})$, since \mathcal{U} extends the Fréchet filter and filters are closed under finite intersections. Furthermore, the sequence $(J_k)_{k \in \mathbb{N}}$ is decreasing with $\bigcap J_k = \emptyset$. For each $i \in \mathbb{N}$, we set $k_i := \max\{k \in \mathbb{N} \mid i \in J_k\}$ and pick a witness s_i for $(\mathfrak{A}, \alpha_i) \models Q(\delta_0 \wedge \dots \wedge \delta_{k_i})$. By Lemma 7.1, we know that there is a $t_{\mathcal{U}} \in Q(\alpha_{\mathcal{U}})$ and $K \in \mathcal{U}$ s.t. $t_{\mathcal{U}} \subseteq \{\gamma_{\mathcal{U}} \mid \gamma_i \in s_i \text{ f.a. } i \in K\}$. Take any $\delta_k \in \Psi$ and $\gamma_{\mathcal{U}} \in t_{\mathcal{U}}$. Then, by construction, for all $i \in J_{k_i} \cap K$, $(\mathfrak{A}, \gamma_i) \models \delta_k$. As $J_{k_i} \cap K \in \mathcal{U}$, $t_{\mathcal{U}}$ realises Ψ .

Similarly, let $\Psi = \{\delta_j \mid j \in \mathbb{N}\}$ be an $s_{\mathcal{U}}$ -type of \mathfrak{A}^* and consider for all $k \in \mathbb{N}$,

$$J_k := \{i \in \mathbb{N} \mid \text{ex. } \gamma_i \in s_i \text{ s.t. } A, \gamma_i \models \delta_0 \wedge \dots \wedge \delta_k\} \cap \{i \in \mathbb{N} \mid i \geq k\}.$$

For $i \in \mathbb{N}$, set $k_i := \max\{k \in \mathbb{N} \mid i \in J_k\}$ and iteratively pick $\gamma_i \in s_i$ for which $\mathfrak{A}, \gamma_i \models \delta_0 \wedge \dots \wedge \delta_{k_i}$. Then $\gamma_{\mathcal{U}}$ realises $s_{\mathcal{U}}$ as desired. \square

Towards the maximality claim in our Lindström result, we use the natural notion of $\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariance.

Definition 7.3 ($\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariance). Let \mathcal{L} be an abstract k -logic and $\mathcal{L}^{\mathcal{Q}}$ a k -quantifier logic. A formula $\varphi \in \mathcal{L}(\sigma)$ is $\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariant if for all k -pointed σ -structures $(\mathfrak{A}, \alpha), (\mathfrak{B}, \beta)$,

$$(\mathfrak{A}, \alpha) \sim_{\mathcal{L}^{\mathcal{Q}}} (\mathfrak{B}, \beta) \implies (\mathfrak{A}, \alpha \models \varphi \iff \mathfrak{B}, \beta \models \varphi).$$

An abstract k -logic \mathcal{L} is $\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariant if all its formulae are, i.e. $\sim_{\mathcal{L}^{\mathcal{Q}}}$ implies $\equiv_{\mathcal{L}}$.

Finally, we obtain a Lindström theorem that characterises $\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariant k -quantifier logics that have the Łoś property.

Theorem 7.4 (a Lindström theorem). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic that has the Łoś property. Then every $\mathcal{L}^{\mathcal{Q}}$ -bisimulation invariant abstract k -logic \mathcal{L} that has the Łoś property is a fragment of $\mathcal{L}^{\mathcal{Q}}$.*

Proof. Let \mathcal{L} be such an abstract k -logic with $\mathcal{L} \not\equiv \mathcal{L}^{\mathcal{Q}}$ and $\varphi \in \mathcal{L}(\sigma)$ be any formula not expressible in $\mathcal{L}^{\mathcal{Q}}$. Since \mathcal{L} has the finite occurrence property, we may assume that $\mathcal{L}^{\mathcal{Q}}(\sigma)$ is countable as \mathcal{Q} is countable by definition and $\varphi \in \mathcal{L}(\sigma_0)$ for some finite $\sigma_0 \subseteq \sigma$. Define the logic $\hat{\mathcal{L}}$ as the FO-closure of $\mathcal{L} \cup \mathcal{L}^{\mathcal{Q}}$. By Lemma 6.4, $\hat{\mathcal{L}}$ has the Łoś property and is thus compact. Hence, we can apply Lemma 6.6 to obtain two pointed σ -structures (\mathfrak{A}, α) and (\mathfrak{B}, β) that have the same $\mathcal{L}^{\mathcal{Q}}$ -theory, while $\mathfrak{A}, \alpha \models \varphi$ and $\mathfrak{B}, \beta \models \neg\varphi$.

Using Lemma 7.2, we find $\mathcal{L}^{\mathcal{Q}}$ -saturated \mathcal{L} -elementary extensions \mathfrak{A}^* and \mathfrak{B}^* of \mathfrak{A} and \mathfrak{B} , respectively. As \mathfrak{A}^* and \mathfrak{B}^* are $\mathcal{L}^{\mathcal{Q}}$ -saturated and (\mathfrak{A}^*, α) and (\mathfrak{B}^*, β) have the same $\mathcal{L}^{\mathcal{Q}}$ -theory, Theorem 5.2 implies that the pointed structures are $\mathcal{L}^{\mathcal{Q}}$ -bisimilar, so that $\varphi \in \mathcal{L}$ would violate the $\mathcal{L}^{\mathcal{Q}}$ -bisimulation invariance of \mathcal{L} as $\mathfrak{A}^*, \alpha \models \varphi$, while $\mathfrak{B}^*, \beta \models \neg\varphi$. \square

By Corollary 6.7, there is essentially no difference between being a fragment of FO and having the Łoś property. Thus, every Lindström theorem w.r.t. the Łoś property is also a characterisation theorem, and we obtain the following different perspective on Theorem 7.4.

Corollary 7.5 (characterisation theorem). *Let $\mathcal{L}^{\mathcal{Q}}$ be a k -quantifier logic that has the Łoś property. Then any $\sim_{\mathcal{L}^{\mathcal{Q}}}$ -invariant formula of FO^k is equivalently expressible in $\mathcal{L}^{\mathcal{Q}}$.*

8 Summary and Outlook

This note suggests a view of a certain kind of quantification that is limited to dealing with k -tuples of elements but takes into account a built-in notion of accessibility of families of sets of k -tuples from given k -tuples. The underlying pattern is thus reminiscent of modal scenarios (broadly conceived) with a twist towards the involvement of second-order objects. While similar notions have been explored in neighbourhood semantics as well as in inquisitive modal logic, our k -quantifiers are in a sense more general in terms of the actual semantics of concrete quantifiers – but also more rigid in terms of the semantics of each individual k -quantifier as the isomorphism type of the underlying plain first-order structure fully determines the accessibility pattern that defines its semantics over that structure. The Ehrenfeucht–Fraïssé style notions of back&forth equivalence associated with a fixed supply of such quantifiers works out naturally as expected. For the associated notions of saturation and the natural Hennessy–Milner theorem to go with it, we here resort to the very strong assumption of compatibility with ultraproducts. By a much deeper theorem of Keisler and Shelah this assumption of course actually ties the expressive power of the logics under consideration to within first-order. Our Lindström result – as is the main target of these investigations – for such k -quantifier logics in the context of abstract k -logics is, however, established on a much more pedestrian route that illustrates the natural involvement of compactness arguments and the rôle of suitable saturation properties in a much more explicit manner. It remains open to which extent weaker assumptions, like, e.g. compactness together with suitable adaptations of a Tarski union property, could be applied to a similar effect against the extremely general background assumptions about abstract k -logics proposed here. Another promising direction for related future research could therefore also concern meaningful limitations of this generality, maybe towards a focus that brings us closer again to more familiar scenarios like neighbourhood or inquisitive semantics.

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