SOLUTION TO THE HENCKELL–RHODES PROBLEM: FINITE F-INVERSE COVERS DO EXIST

K. AUINGER, J. BITTERLICH AND M. OTTO

Abstract. For a finite connected graph $\mathcal{E}$ with set of edges E, a finite E-generated group G is constructed such that the set of relations $p = 1$ satisfied by G (with p a word over $E \cup E^{-1}$) is closed under deletion of generators (i.e. edges). As a consequence, every element $g \in G$ admits a unique minimal set of edges needed to represent g (the *content* of g). The crucial property of the group G is that the connectivity in the graph $\mathcal{E}$ is encoded in G in the following sense: if a word p forms a path $u \longrightarrow v$ in $\mathcal{E}$ then the content of the corresponding group element $[p]_G$ spans a connected subgraph of $\mathcal{E}$ containing the vertices u and v. As an application it is shown that every finite inverse monoid admits a finite F-inverse cover. This solves a long-standing problem of Henckell and Rhodes.

1. Introduction

In the influential paper [11], Henckell and Rhodes stated a series of conjectures and two problems. The paper was concerned with the celebrated question whether every finite block group M (a monoid in which every von Neumann regular element admits a unique inverse) is a quotient of a submonoid of the power monoid $\mathbb{P}(G)$ of some finite group G. The authors presented an affirmative answer to the question modulo some conjecture, namely about the structure of pointlike sets; a subset X of a finite monoid M is *pointlike* (with respect to groups) if and only if in *every* subdirect product $T \subseteq M \times G$ of M with a finite group G there exists an element $g \in G$ with $X \times \{g\} \subseteq T$ (that is, all elements of X *relate* to some point $g \in G$.) All stated conjectures concerned various aspects of pointlike sets. For example, for inverse monoids M the conjecture was that a subset of M is pointlike if and only if it admits an upper bound with respect to the natural partial order of M. Shortly after, all stated conjectures and one of the two problems (about *liftable tuples* of monoids) have been verified respectively solved by Ash in his celebrated paper [3]. The importance of the latter paper went beyond its immediate task as in the following years interesting and deep connections with the profinite topology of the free group [21] and

2010 *Mathematics Subject Classification.* 20M18, 20B25, 05C25.

Key words and phrases. F-inverse monoid, F-inverse cover, permutation group, action graph, Cayley graph.

finite model theory [12] have been revealed and studied [1, 2]. Yet the second stated problem which was called by the authors a "stronger form of the pointlike conjecture for inverse monoids" has not been solved in Ash's paper and has since then attracted considerable attention [14, 25, 26, 5, 24, 23, 8]. It asked:

Problem 1.1. Does every finite inverse monoid admit a finite F-inverse cover?

An inverse monoid is *F-inverse* if every congruence class of the smallest group congruence σ of F admits a greatest element (with respect to the natural partial order) and an inverse monoid F is a *cover* of an inverse monoid M if there exists a surjective, idempotent separating homomorphism $F \to M$.

The second author was the first to understand that Problem 1.1 admits a positive solution. In his paper [9] and his dissertation [10] he presented a proof which strongly relied on a result of the third author [17, 18] about the existence of certain finite groupoids. Later, some flaws were discovered in [17, 18] which, however, have been fixed in the meantime [19]. The intention of the present paper is to give a complete and self-contained presentation of the solution to Problem 1.1 (up to classical results on inverse monoids) which is based on the ideas and proofs of [19] but is in a sense tailored for what is needed in the present context, and is presented in a language which (hopefully) makes it easier accessible to the semigroup community. The paper is organised as follows: Section 2 collects prerequisites from inverse monoids, graphs and a proof that the existence of certain groups yields a positive solution of Problem 1.1. Section 3 introduces the main graph theoretic tools while Section 4 presents two crucial technical results. Finally, in Section 5 we present a construction of the required groups. This construction intends to "reflect the geometry" of a given finite graph.

2. Inverse monoids

2.1. **Preliminaries.** A monoid M is *inverse* if every element $x \in M$ admits a unique element x^{-1}, called the *inverse* of x, satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. This gives rise to a unary operation ${}^{-1}\colon M \to M$ and an inverse monoid may equivalently be defined as an algebraic structure $(M; \cdot, {}^{-1}, 1)$ with $\cdot$ an associative binary operation, 1 a neutral element with respect to $\cdot$ and a unary operation ${}^{-1}$ satisfying the laws

$$(x^{-1})^{-1} = x,\ (xy)^{-1} = y^{-1}x^{-1},\ xx^{-1}x = x \text{ and } xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

In particular, the class of all inverse monoids forms a variety of algebraic structures (in the sense of universal algebra), the variety of all groups $(G; \cdot, {}^{-1}, 1)$ being a subvariety. From basic facts of universal algebra it follows that every inverse monoid M admits a smallest congruence such that the corresponding quotient structure is a group. This congruence is usually denoted σ and it can be characterised as the smallest congruence on M that identifies all idempotents of M with each other. Another way to characterise this congruence is this: two elements $x, y \in M$ are σ-related if and only if $xe = ye$ for some idempotent e of M (and this is equivalent to $fx = fy$ for some idempotent f of M).

Every inverse monoid M is equipped with a partial order $\leq$, the *natural order*, defined by $x \leq y$ if and only if $x = ye$ for some idempotent e of M (this is equivalent to $x = fy$ for some idempotent f of M). This order is compatible with the binary operation of M. In terms of the natural order, the congruence σ can be characterised as the smallest congruence for which the natural order on the quotient is the identity relation, and, likewise as the smallest congruence that identifies every pair of $\leq$-comparable elements; the latter leads to yet another description of σ: two elements x and y are σ-related if and only if they admit a common lower bound with respect to $\leq$. For further information on inverse monoids the reader is referred to the monographs by Petrich [20] and Lawson [14].

An inverse monoid is *F-inverse* if every σ-class possesses a greatest element with respect to $\leq$. For recent developments concerning the systematic study of F-inverse monoids and their relevance in various contexts the reader is referred to [7] and the literature cited there. An F-inverse monoid F is an *F-inverse cover* of the inverse monoid M if there exists a surjective idempotent separating homomorphism $F \to M$. As mentioned in the introduction, it has been an outstanding open problem whether every finite inverse monoid M admits a finite F-inverse cover. In order to formulate the following very useful result [20, Theorem VII.6.11] we need the concept of premorphism: for inverse monoids M and N, a mapping $\psi\colon M \to N$ is a *premorphism* if $\psi(1) = 1$, $\psi(m^{-1}) = (\psi m)^{-1}$ and $(\psi m)(\psi n) \leq \psi(mn)$ for all $m, n \in M$.

Theorem 2.1. *Let H be a group and M be an inverse monoid; if $\psi\colon H \to M$ is a premorphism such that, for every $m \in M$, there exists $h \in H$ with $m \leq \psi(h)$, then the subdirect product*

$$F := \{(m, h) \in M \times H \mid m \leq \psi(h)\}$$

is an F-inverse cover of M. Conversely, every F-inverse cover of M can be so constructed.

The following is an easy observation.

Observation 2.2. *Suppose that $\psi\colon H \to M$ is as in Theorem 2.1 and $\pi\colon M \to N$ is a surjective homomorphism with N an inverse monoid; then the composition $\pi \circ \psi\colon H \to N$ is a premorphism which also satisfies the condition of Theorem 2.1.*

Hence the task for Problem 1.1 is, given a finite inverse monoid N, to find a finite group H which admits a premorphism $H \to N$ satisfying the condition of Theorem 2.1. Observation 2.2 eases the situation a bit since we need to do so only for a special type of inverse monoids (in the rôle of M) which we shall describe below (see § 2.4).

Throughout, for any non-empty set X (of letters, of edges, etc.) we let $X^{-1} := \{x^{-1} \mid x \in X\}$ be a disjoint copy of X consisting of formal inverses of the elements of X, and set $\widetilde{X} := X \cup X^{-1}$. The mapping $x \to x^{-1}$ is extended to an involution of $\widetilde{X}$ by setting $(x^{-1})^{-1} = x$, for all $x \in X$. We let $\widetilde{X}^*$ be the free monoid over $\widetilde{X}$, which, subject to $(x_1 \cdots x_n)^{-1} = x_n^{-1} \cdots x_1^{-1}$ (where $x_i \in \widetilde{X}$), is the *free involutory monoid* over X. The elements of $\widetilde{X}^*$ are called *words over* $\widetilde{X}$, and we let 1 denote the *empty word*.

2.2. **Graphs.** In this paper, we consider a very liberal notion of graph structures, admitting multiple directed edges between pairs of vertices, including directed loops at individual vertices. In the literature, such structures are often called *directed multigraphs* or *quivers*. The following formalisation is convenient for our purposes. A *graph* $\mathcal{E}$ is a *two-sorted structure* $(V \cup K; \alpha, \omega, {}^{-1})$ with V its set of *vertices*, K its set of *edges* (disjoint from V), with *incidence functions* $\alpha\colon K \to V$ and $\omega\colon K \to V$, selecting, for each edge e the *initial* vertex αe and the *terminal* vertex ωe, and *involution* ${}^{-1}\colon K \to K$ satisfying $\alpha e = \omega e^{-1}$, $\omega e = \alpha e^{-1}$ and $e \neq e^{-1}$ for every edge $e \in K$. Instead of *initial/terminal vertex* the terms *source/target* are also used in the literature. One should think of an edge e with $\alpha e = u$ and $\omega e = v$ in "geometric" terms as $e\colon \underset{u}{\bullet}\!\longrightarrow\!\underset{v}{\bullet}$ and its inverse $e^{-1}\colon \underset{u}{\bullet}\!\longleftarrow\!\underset{v}{\bullet}$ as "the same edge but traversed in the opposite direction". A graph $(V \cup K; \alpha, \omega, {}^{-1})$ is *oriented* if the edge set K is partitioned as $K = E \cup E^{-1} = \widetilde{E}$ such that every ${}^{-1}$-orbit contains exactly one element of E and one of E^{-1}; the edges in E are the *positive* or *positively oriented* edges, those in E^{-1} the *negative* or *negatively oriented* ones. An oriented graph $\mathcal{E}$ with set of positive edges E will be denoted as $\mathcal{E} = (V \cup \widetilde{E}; \alpha, \omega, {}^{-1})$.

A *subgraph* of the graph $\mathcal{E}$ is a substructure that is induced over a subset of $V \cup K$ which is closed under the operations α and ${}^{-1}$ (and therefore also under ω). In particular, every subset $S \subseteq V \cup K$ generates a unique minimal subgraph $\langle S \rangle$ of $\mathcal{E}$ containing S, which is the *subgraph of* $\mathcal{E}$ *spanned by* S. An *automorphism* of a graph $\mathcal{E} = (V \cup K; \alpha, \omega, {}^{-1})$ is a map $\varphi = \varphi_V \cup \varphi_K\colon V \cup K \to V \cup K$ with $\varphi_V\colon V \to V$, $\varphi_K\colon K \to K$ being bijections satisfying for all $e \in K$:

$$\alpha\varphi_K(e) = \varphi_V(\alpha e),\ \omega\varphi_K(e) = \varphi_V(\omega e),\ \varphi_K(e^{-1}) = (\varphi_K(e))^{-1}.$$

We note that the second equality is a consequence of the first and third. In the oriented case we require in addition that $\varphi_{\widetilde{E}}(E) = E$ and (therefore also) $\varphi_{\widetilde{E}}(E^{-1}) = E^{-1}$. A benefit from our definition of a graph as a two-sorted functional rather than a relational structure is that there is no distinction between weak and induced subgraphs and that concepts like quotient and homomorphism are easier to handle.

Let A be a finite set; a *labelling* of the graph $\mathcal{E} = (V \cup K; \alpha, \omega, {}^{-1})$ *by the alphabet* A (an A-*labelling*, for short) is a mapping $\ell\colon K \to \widetilde{A}$ respecting the involution: $\ell(e^{-1}) = \ell(e)^{-1}$ for all $e \in K$. The labelling $\ell\colon K \to \widetilde{A}$ gives rise to an orientation of $\mathcal{E}$: setting $E := \{e \in K \mid \ell(e) \in A\}$ (positive edges) and $E^{-1} := \{e \in K \mid \ell(e) \in A^{-1}\}$ (negative edges) then $E \cap E^{-1} = \varnothing$ and we get $K = \widetilde{E}$.

We consider A-labelled graphs as structures $(V \cup K; \alpha, \omega, {}^{-1}, \ell, A)$ in their own right. By a *subgraph of an A-labelled graph* we mean just a subgraph with the induced labelling. Morphisms of A-labelled graphs are naturally defined as follows. Let $\mathcal{K} = (V \cup K; \alpha, \omega, {}^{-1}, \ell, A)$ and $\mathcal{L} = (W \cup L; \alpha, \omega, {}^{-1}, \ell, A)$ be A-labelled graphs. *A morphism $\varphi \colon \mathcal{K} \to \mathcal{L}$ of A-labelled graphs* is a mapping $\varphi \colon V \cup K \to W \cup L$, mapping vertices to vertices and edges to edges, that is compatible with the operations α and ${}^{-1}$ (and therefore also ω) as well as with the labelling.

A *congruence Θ on the A-labelled graph* $\mathcal{K} = (V \cup K; \alpha, \omega, {}^{-1}, \ell, A)$ is an equivalence relation on $V \cup K$ contained in $(V \times V) \cup (K \times K)$ which is compatible with the operations α and ${}^{-1}$ (therefore also ω) and respects ℓ:

$$e \;\Theta\; f \Longrightarrow \alpha e \;\Theta\; \alpha f,\ \omega e \;\Theta\; \omega f,\ e^{-1} \;\Theta\; f^{-1} \text{ for all } e, f \in K$$

and

$$e \;\Theta\; f \Longrightarrow \ell(e) = \ell(f) \text{ for all } e, f \in K.$$

The definition of the quotient graph $\mathcal{K}/\Theta$ for a congruence Θ is obvious, and we have the usual Homomorphism Theorem.

A *non-empty path* π in $\mathcal{E}$ is a sequence $\pi = e_1 e_2 \cdots e_n$ $(n \geq 1)$ of *consecutive* edges (that is $\omega e_i = \alpha e_{i+1}$ for all $1 \leq i < n$); we set $\alpha\pi := \alpha e_1$ and $\omega\pi = \omega e_n$ (denoting the initial and terminal vertices of the path π); the *inverse path* π^{-1} is the path $\pi^{-1} := e_n^{-1} \cdots e_1^{-1}$; it has initial vertex $\alpha\pi^{-1} = \omega\pi$ and terminal vertex $\omega\pi^{-1} = \alpha\pi$. We also consider, for each vertex v, the *empty path at v*, denoted ε_v for which we set $\alpha\varepsilon_v = v = \omega\varepsilon_v$ and $\varepsilon_v^{-1} = \varepsilon_v$ (it is convenient to view ε_v as the "empty edge at v" and one may identify this with the vertex v itself). We say that π is a path *from $u = \alpha\pi$ to $v = \omega\pi$*, and we will also say that u and v are *connected by* π (and likewise by π^{-1}). A graph is connected if any two vertices can be connected by some path. The subgraph $\langle\pi\rangle$ spanned by the non-empty path π is the graph spanned by the edges of π, it coincides with $\langle\pi^{-1}\rangle$; the graph spanned by an empty path ε_v simply is $\{v\}$ (one vertex, no edge). For a path $e_1 \cdots e_k$ in an A-labelled graph $\mathcal{E}$, its label is $\ell(e_1 \cdots e_k) := \ell(e_1) \cdots \ell(e_k)$ which is a word in $\widetilde{A}^*$.

2.3. ***A*-generated inverse monoids.** We fix a non-empty set A (called alphabet in this context). An inverse monoid M together with a (not necessarily injective) mapping $i_M \colon A \to M$ (called *assignment function*) is an *A-generated* inverse monoid if M is generated by $i_M(A)$, as an inverse monoid, that is, generated with respect to the operations $1, \cdot, {}^{-1}$. For every congruence ρ of an A-generated inverse monoid M, the quotient M/ρ is A-generated with respect to the map $i_{M/\rho} = \pi_\rho \circ i_M$ where π_ρ is the projection $M \to M/\rho$. A *morphism ψ from the A-generated inverse monoid M to the A-generated inverse monoid* N is a homomorphism $M \to N$ *respecting* *generators from* A, that is, satisfying $i_N = \psi \circ i_M$. If it exists, such a morphism is unique and surjective and is called *canonical morphism*, denoted $\psi\colon M \twoheadrightarrow N$. In this situation, M is an *expansion of* N. The special case of A-generated groups will play a significant rôle in this paper.

As already mentioned, the assignment function is not necessarily injective, and, what is more, some generators may even be sent to the identity element of M. This is not a deficiency, but rather is adequate in our context, since we want the quotient of an A-generated structure to be again A-generated. In particular M/σ, the quotient of an A-generated inverse monoid M modulo the smallest group congruence σ, is an A-generated group.

The assignment function i_M is usually not explicitly mentioned; it uniquely extends to a homomorphism $[\ \ ]_M\colon \widetilde{A}^* \to M$ (of involutory monoids). For every word $p \in \widetilde{A}^*$, $[p]_M$ is the *value of* p *in* M. For two words $p, q \in \widetilde{A}^*$, the A-generated inverse monoid M *satisfies the relation* $p = q$ if $[p]_M = [q]_M$ and M *avoids the relation* $p = q$ if $[p]_M \neq [q]_M$.

2.4. **Cayley graphs of A-generated groups and the Margolis–Meakin expansion.** Given an A-generated group Q we define the *Cayley graph* $\mathcal{Q}$ of Q by the following data; as an A-labelled graph, this graph $\mathcal{Q}$ depends on the underlying assignment function i_Q:

- the set of vertices of $\mathcal{Q}$ is Q,
- the set of edges of $\mathcal{Q}$ is $Q \times \widetilde{A}$, and, for $g \in Q,\ a \in \widetilde{A}$, the incidence functions, involution and labelling are defined according to

$$\begin{aligned}
\alpha(g,a) &:= g,\\
\omega(g,a) &:= g[a]_Q,\\
(g,a)^{-1} &:= (g[a]_Q, a^{-1}),\\
\ell(g,a) &:= a.
\end{aligned}$$

The edge (g,a) should be thought of as $\underset{g}{\bullet}\xrightarrow{\ a\ }\underset{ga}{\bullet}$, its inverse as $\underset{g}{\bullet}\xleftarrow{\ a^{-1}\ }\underset{ga}{\bullet}$, where ga stands for $g[a]_Q$. We note that Q acts on $\mathcal{Q}$ by left multiplication as a group of automorphisms via

$$g \longmapsto {}^h g := hg \quad \text{and} \quad (g,a) \longmapsto {}^h(g,a) := (hg,a)$$

for all $g, h \in Q$ and $(g,a) \in Q \times \widetilde{A}$, where h is an element of the acting group Q, g a vertex of $\mathcal{Q}$ and (g,a) an edge of $\mathcal{Q}$.

We arrive at the important concept of the *Margolis–Meakin expansion* $M(Q)$ of an A-generated group Q [15]. For a given A-generated group Q, the Margolis–Meakin expansion $M(Q)$ consists of all pairs $(\mathcal{K}, g)$ with $g \in Q$ and $\mathcal{K}$ a finite connected subgraph of the Cayley graph $\mathcal{Q}$ of Q containing the vertices 1 and g. Endowed with the multiplication

$$(\mathcal{K}, g)(\mathcal{L}, h) = (\mathcal{K} \cup {}^g\mathcal{L}, gh)$$

and involution

$$(\mathcal{K}, g)^{-1} = ({}^{g^{-1}}\mathcal{K}, g^{-1})$$

the set $M(Q)$ becomes an A-generated inverse monoid with identity element $(\{1\}, 1)$ and with respect to the assignment function

$$A \to M(Q), \quad a \mapsto (\langle(1, a)\rangle, [a]_Q).$$

The value of some word $p \in \widetilde{A}^*$ in $M(Q)$ is

$$[p]_{M(Q)} = (\langle \pi_1^{\mathcal{Q}}(p)\rangle, [p]_Q),$$

where $\pi_1^{\mathcal{Q}}(p)$ is the path in $\mathcal{Q}$ starting at 1 and having label p; the natural partial order on $M(Q)$ is given by

$$(\mathcal{K}, g) \le (\mathcal{L}, h) \text{ if and only if } \mathcal{K} \supseteq \mathcal{L} \text{ and } g = h.$$

The Margolis–Meakin expansion plays an important rôle in the theory of inverse semigroups; for its universal property the reader is referred to [15] or [7]. Most relevant for our purpose is the following, which is a consequence of the results of [15].

Theorem 2.3. *Every finite inverse monoid M arises as a quotient of the Margolis–Meakin expansion $M(Q)$ of some finite A-generated group Q for some finite alphabet A.*

Consequently, in order to find, for a finite inverse monoid M, a finite group H with a premorphism $\psi\colon H \to M$ satisfying the condition of Theorem 2.1, according to Observation 2.2 it is sufficient to do so for M being the Margolis–Meakin expansion $M(Q)$ of any finite A-generated group Q.

2.5. ***F*-inverse covers.** For a given A-generated group Q as above, we now seek to provide an expansion H of Q, which will allow us to use Theorems 2.1 and 2.3 together with Observation 2.2 towards the construction of F-inverse covers, as in Theorem 2.7 below. First we isolate an important property of groups generated by an alphabet.

Definition 2.4 (X-generated group with content function)**.** Let X be any alphabet; an X-generated group R *has a content function* C if for every element $g \in R$ there is a unique $\subseteq$-minimal subset $\mathrm{C}(g)$ of X such that g is represented as a product of elements of $\mathrm{C}(g)$ and their inverses.

We need to define one further property, which will be crucial towards the construction of a group H admitting a premorphism $\psi\colon H \to M(Q)$, to the Margolis–Meakin expansion $M(Q)$ satisfying the condition of Theorem 2.1.

Definition 2.5 (group reflecting the structure of a Cayley graph)**.** Let Q be an A-generated group with Cayley graph $\mathcal{Q}$ and let $E = Q \times A$ be the set of positive edges of $\mathcal{Q}$. An E-generated group G *reflects the structure of* $\mathcal{Q}$ if the following hold.

(1) The action of Q on E by left multiplication extends to an action of Q on G by automorphisms on the left (denoted $(g, \xi) \mapsto {}^{g}\xi$ for $g \in Q$ and $\xi \in G$).
(2) G has a content function C such that, for any $p \in \widetilde{E}^*$ which forms a path $g \longrightarrow h$ in $\mathcal{Q}$ the following hold:
  (a) if $\mathrm{C}([p]_G) = \varnothing$, that is if $[p]_G = 1$, then $g = h$,
  (b) if $\mathrm{C}([p]_G) \neq \varnothing$, that is if $[p]_G \neq 1$, then the content $\mathrm{C}([p]_G)$ spans a connected subgraph of $\mathcal{Q}$ containing g and h.

Next let Q be an A-generated group and, for $E = Q \times A$, let G be an E-generated group which reflects the structure of the Cayley graph $\mathcal{Q}$ of Q. Since Q acts on G by automorphisms on the left, we can form the semidirect product $G \rtimes Q$, which consists of the set $G \times Q$ endowed with the binary operation

$$(\gamma, g)(\eta, h) := (\gamma \cdot {}^{g}\eta, gh),$$

inversion

$$(\gamma, g)^{-1} := ({}^{g^{-1}}\gamma^{-1}, g^{-1})$$

and identity element $(1_G, 1_Q)$. Consider the following A-generated subgroup H of $G \rtimes Q$

$$H := \langle ([(1, a)]_G, [a]_Q) \mid a \in A \rangle \leq G \rtimes Q. \tag{2.1}$$

Similarly to the value $[p]_{M(Q)}$ mentioned above one has that for a word $p \in \widetilde{A}^*$ the value of p in H is

$$[p]_H = ([\pi_1^{\mathcal{Q}}(p)]_G, [p]_Q) \tag{2.2}$$

where, again, $\pi_1^{\mathcal{Q}}(p)$ is the unique path in $\mathcal{Q}$ starting at 1 and being labelled p, interpreted as a word over $\widetilde{E}$. In particular, H is an expansion of Q with canonical morphism $([\pi_1^{\mathcal{Q}}(p)]_G, [p]_Q) \mapsto [p]_Q$.

Theorem 2.6. *Let Q be an A-generated group, for $E = Q \times A$ let G be an E-generated group (with content function* C*) which reflects the structure of the Cayley graph $\mathcal{Q}$ of Q (Definition 2.5), and let H be the group defined by* (2.1)*. Then the mapping*

$$\psi \colon H \to M(Q), \quad (\gamma, g) \mapsto \begin{cases} (\{1_Q\}, 1_Q) \ \textit{if } (\gamma, g) = (1_G, 1_Q) \\ (\langle \mathrm{C}(\gamma) \rangle, g) \ \textit{if } (\gamma, g) \neq (1_G, 1_Q) \end{cases}$$

is a premorphism which satisfies the condition formulated in Theorem 2.1.

Proof. Recall that for $\gamma \in G$, $\mathrm{C}(\gamma) = \varnothing$ if and only if $\gamma = 1_G$. The definition of ψ makes sense only if $(1_G, g) \in H$ implies $g = 1_Q$. Let $p \in \widetilde{A}^*$ be such that $[p]_H = (1_G, g)$; then $1_G = [\pi_1^{\mathcal{Q}}(p)]_G$ and $\pi_1^{\mathcal{Q}}(p)$ is the path in $\mathcal{Q}$ starting at 1_Q and being labelled p. By (2.2) the terminal vertex of this path is $[p]_Q = g$; but from Definition 2.5 (2a) it follows that this path is closed, hence $[p]_Q = 1_Q$. So, if $[p]_H \neq 1_H$ then $[\pi_1^{\mathcal{Q}}(p)]_G \neq 1_G$ and, by Definition 2.5 (2b), the content $C([\pi_1^{\mathcal{Q}}(p)]_G)$ spans a connected subgraph of $\mathcal{Q}$ containing 1_Q and

$[p]_Q$ so that $\psi([p]_H) \in M(Q)$, as required. Definition 2.5, $\mathrm{C}(\gamma^{-1}) = \mathrm{C}(\gamma)$, $\mathrm{C}({}^g\gamma) = {}^g\mathrm{C}(\gamma)$ and $\mathrm{C}(\gamma \cdot \eta) \subseteq \mathrm{C}(\gamma) \cup \mathrm{C}(\eta)$ imply that ψ is a premorphism.

Finally, let $(\mathcal{K}, g) \in M(Q)$ and $p \in \widetilde{A}^*$ be such that $[p]_{M(Q)} = (\mathcal{K}, g)$. Then $\mathcal{K} = \langle \pi_1^{\mathcal{Q}}(p) \rangle$, $g = [p]_Q$ and $[p]_H = (\gamma, g)$ where $\gamma = [\pi_1^{\mathcal{Q}}(p)]_G$ (the G-value of the path $\pi_1^{\mathcal{Q}}(p)$). If $\mathrm{C}(\gamma) = \varnothing$, that is, $\gamma = 1_G$ then $g = 1_Q$, hence $\psi(\gamma, g) = (\{1_Q\}, 1_Q) \geq (\mathcal{K}, 1_Q) = (\mathcal{K}, g)$. Otherwise, if $\mathrm{C}(\gamma) \neq \varnothing$ then every edge of $\mathrm{C}([\pi_1^{\mathcal{Q}}(p)]_G)$ belongs to $\langle \pi_1^{\mathcal{Q}}(p) \rangle$, hence

$$\langle \mathrm{C}(\gamma) \rangle = \langle \mathrm{C}([\pi_1^{\mathcal{Q}}(p)]_G) \rangle \subseteq \langle \pi_1^{\mathcal{Q}}(p) \rangle,$$

so that $\psi(\gamma, g) = (\langle \mathrm{C}(\gamma) \rangle, g) \geq (\mathcal{K}, g)$, as required. $\square$

Theorem 2.1 now implies that an F-inverse cover of $M(Q)$ can be constructed as a subdirect product of H with $M(Q)$. Observation 2.2 in combination with Theorem 2.3 implies the result promised in the title of the paper.

Theorem 2.7. *Every finite inverse monoid admits a finite F-inverse cover.*

In order to prove this theorem it is sufficient to construct, for any finite A-generated group Q and $E = Q \times A$ a finite E-generated group G which reflects the structure of the Cayley graph $\mathcal{Q}$ of Q according to Definition 2.5. The existence of such a group G is guaranteed by the following more general lemma, which is the main result of the paper. For item (1) recall that every automorphism of an oriented graph induces a permutation of its set of positive edges.

Lemma 2.8 (main lemma)**.** *For every finite connected oriented graph $\mathcal{E} = (V \cup \widetilde{E}; \alpha, \omega, {}^{-1})$ there exists a finite E-generated group G which has the following properties:*

(1) *Every permutation of E induced by an automorphism of $\mathcal{E}$ extends to an automorphism of G.*
(2) *The set of relations $p = 1$ satisfied by G (with $p \in \widetilde{E}^*$) is closed under the deletion of generators and thus G has a content function* C *(Proposition 3.1).*
(3) *For any word $p \in \widetilde{E}^*$ which forms a path $u \longrightarrow v$ in $\mathcal{E}$ (with u and v not necessarily distinct vertices of $\mathcal{E}$) the following hold:*
    (a) *if $\mathrm{C}([p]_G) = \varnothing$ then $u = v$,*
    (b) *if $\mathrm{C}([p]_G) \neq \varnothing$ then $\mathrm{C}([p]_G)$ spans a connected subgraph of $\mathcal{E}$ containing the vertices u and v.*

The remainder of the paper is devoted to proving Lemma 2.8. This requires quite a bit of work. It will be accomplished in Section 5. In order to achieve this goal we introduce several graph-theoretic constructions which will be presented in Sections 3 and 4. The results in these two sections are of a more general nature and are of independent interest.

3. Tools

In this section we introduce some graph theoretic constructions which later will enable the construction of a group G as mentioned above. The group itself will be realised as a permutation group defined by its *action graph*. It is a well-established approach to construct finite A-generated groups which avoid certain unwanted relations to proceed as described in the following. First encode the relations in a finite A-labelled directed graph $\mathcal{X}$ — the set of unwanted relations will usually be infinite, but must in some sense be regular (recognisable by a finite automaton). Next take a quotient $\mathcal{X}/{\equiv}$ of $\mathcal{X}$ which guarantees that the edge labels from A induce partial injective mappings on the vertex set. Finally form some *completion* $\overline{\mathcal{X}/{\equiv}}$ of $\mathcal{X}/{\equiv}$, through extending the partial injective mappings to total permutations of the vertex set of $\mathcal{X}/{\equiv}$ or of some finite superset. The letters $a \in A$ then act as permutations on the finite set of vertices of $\overline{\mathcal{X}/{\equiv}}$ and one gets a finite permutation group that avoids the unwanted relations. A meanwhile classical application of this approach is Stallings' proof of Hall's Theorem that every finitely generated subgroup of a free group F is closed in the profinite topology of F [22]. Many more examples can be found in [13, 6] and elsewhere. In his paper [3] Ash has definitely developed some mastership of arguments of this kind. Independently, the third author has suggested a considerable refinement of this approach [16]. He proposed a construction which is inductive on the subsets of the generating set A in the sense that the kth group G_k satisfies/avoids all relations $p = 1$ in at most k letters that should be satisfied/avoided by the final group G. In the step $G_k \rightsquigarrow G_{k+1}$ not only new relations $p = 1$ in more than k letters are added which are to be avoided (by adding components to the graph which defines G_k) but, at the same time, the relations in at most k letters must be preserved. The motivation for this approach has come from some relevant applications to hypergraph coverings and model theory [16]. The constructions in this section and the results of the next section are of this flavour and are taken from the third author's [19].

3.1. **E-graphs and E-groups.** We slightly change perspective: since the edges of the graph $\mathcal{E}$ of Lemma 2.8 are the letters of the labelling alphabet we now denote the labelling alphabet by E. An E-labelled graph is *folded* [13] or an *E-graph* if every vertex u has, for every label $a \in \widetilde{E}$, at most one edge with initial vertex u and label a. In an E-graph $\mathcal{K}$, for every word $p \in \widetilde{E}^*$ and every vertex u there is at most one path $\pi = \pi_u^{\mathcal{K}}$ with initial vertex $\alpha\pi = u$ and label $\ell(\pi) = p$. For a path π in $\mathcal{K}$ with initial vertex u, terminal vertex v and label $p \in \widetilde{A}^*$ (for $A \subseteq E$) we write $u \xrightarrow{p} v$ or $p\colon u \longrightarrow v$ and we call π an *A-path* $u \longrightarrow v$; the vertices u and v are *A-connected* in $\mathcal{K}$. The *A-component* of a vertex v of the E-graph $\mathcal{K}$, denoted $v\mathcal{K}[A]$, is the subgraph of $\mathcal{K}$ spanned by all paths in $\mathcal{K}$ having initial vertex v and whose labels are in $\widetilde{A}^*$. A labelled graph $\mathcal{K}$ is called *complete* or a *(group) action graph* (also called *permutation automaton*) if every vertex u has, for every label $a \in \widetilde{E}$ exactly one edge f with initial vertex $\alpha f = u$ and label $\ell(f) = a$; in this case, for every word $p \in \widetilde{E}^*$ and every vertex u there exists exactly one path $\pi = \pi_u^{\mathcal{K}}(p)$ starting at u and having label p. We set $u \cdot p := \omega(\pi_u^{\mathcal{K}}(p))$, the terminal vertex of the path starting at u and being labelled p; then, for every $p \in \widetilde{E}^*$, the mapping $u \mapsto u \cdot p$ is a permutation $[p]$ of the set V of vertices of $\mathcal{K}$. Thus the involutory monoid $\widetilde{E}^*$ acts on V by permutations on the right. The permutation group

$$\mathscr{T}(\mathcal{K}) := \{[p] \mid p \in \widetilde{E}^*\} \tag{3.1}$$

obtained this way, is called the *transition group* $\mathscr{T}(\mathcal{K})$ of the graph $\mathcal{K}$. This transition group $\mathscr{T}(\mathcal{K})$ is an E-generated group in a natural way, the letter $e \in \widetilde{E}$ induces the permutation $[e]$ which maps every vertex u to the terminal vertex $\omega\pi_u(e)$ of the edge $\pi_u(e)$ which is the unique edge with initial vertex u and label e. Note that this edge may be a loop edge for every vertex u (so $[e]$ might be the identity element of $\mathscr{T}(\mathcal{K})$). Moreover, it may happen that distinct letters $e \neq f \in E$ induce the same permutation.

A crucial fact concerning the transition group $G = \mathscr{T}(\mathcal{K})$ is the following: for every connected component $\mathcal{C}$ of $\mathcal{K}$ and every vertex u of $\mathcal{C}$ there is a unique surjective graph morphism $\varphi_u \colon \mathcal{G} \twoheadrightarrow \mathcal{C}$ for which $\varphi_u(1) = u$ ($\mathcal{G}$ the Cayley graph of G with 1 the identity vertex); we call φ_u the *canonical morphism* $\mathcal{G} \twoheadrightarrow \mathcal{C}$ *with respect to* u; occasionally we shall leave the vertex u undetermined and shall speak of *some* canonical morphism $\mathcal{G} \twoheadrightarrow \mathcal{C}$.

An E-graph is *weakly complete* if, for every letter $a \in \widetilde{E}$, the partial injective mapping on V induced by a is a permutation on its domain; in other words, provided that the graph is finite, the subgraph spanned by all edges with label a is a disjoint union of cycle graphs (a-cycles). For every weakly complete graph $\mathcal{K}$ we denote by $\overline{\mathcal{K}}$ its *trivial completion*, that is, the complete graph obtained by adding, for every $a \in \widetilde{E}$, a loop edge with label a to every vertex not already contained in an a-cycle of $\mathcal{K}$.

3.2. k-retractable groups, content function and k-stable expansions.

For $a \in E$ and $p \in \widetilde{E}^*$ let $p_{a\to 1}$ be the word obtained from p by deletion of all occurrences of a and a^{-1} in p. Let G be an E-group; for every $A \subseteq E$ let $G[A]$ be the A-generated subgroup of G. An E-group G is *retractable* if, for all words $p, q \in \widetilde{E}^*$ and every letter $a \in E$ the following holds:[1]

$$[p]_G = [q]_G \Longrightarrow [p_{a\to 1}]_G = [q_{a\to 1}]_G.$$

This definition means that for every subset $A \subseteq E$ the mapping

$$E \to E \cup \{1\},\ a \mapsto \begin{cases} a \text{ if } a \in A \\ 1 \text{ if } a \notin A \end{cases}$$

[1] It suffices to restrict this postulate to the case $q = 1$.

extends to an endomorphism ψ_A of G, which in fact is a retract endomorphism onto $G[A]$ (the image of ψ_A is $G[A]$ and its restriction to $G[A]$ is the identity mapping). Moreover, G is A-*retractable* if $G[A]$ is retractable (as an A-group), and, for $k \leq |E|$, G is k-*retractable* if G is A-retractable for every $A \subseteq E$ with $|A| = k$. Of course, k-retractability implies l-retractability for all $l \leq k$, and every group is 1-retractable.

For a word $p \in \widetilde{E}^*$ the *content* $\mathrm{co}(p)$ is the set of all letters $a \in E$ for which a or a^{-1} occurs in p. The importance of retractable E-groups for our purpose comes from the fact that such E-groups admit a *content function.* Indeed, assume that G is retractable. Then, for $p, q \in \widetilde{E}^*$ and $a \in E$ the equality $[p]_G = [q]_G$ implies $[p_{a\to 1}]_G = [q_{a\to 1}]_G$. Suppose now that $a \in \mathrm{co}(p)$ but $a \notin \mathrm{co}(q)$. Then the words q and $q_{a\to 1}$ are identical. Hence $[p]_G = [q]_G$ implies

$$[p_{a\to 1}]_G = [q_{a\to 1}]_G = [q]_G = [p]_G.$$

In this way, we may delete (without changing its value $[p]_G$) every letter in a word p which does not occur in every other representation q of the group element $[p]_G$. The set of all letters of $\mathrm{co}(p)$ which cannot be deleted this way is the *content* $\mathrm{C}([p]_G)$ of the group element $[p]_G$.

Proposition 3.1. *Every retractable group has a content function.*

In case G is retractable, for any two subsets $A, B \subseteq E$ we have

$$G[A] \cap G[B] = G[A \cap B]. \tag{3.2}$$

Groups satisfying this condition for all $A, B \subseteq E$ have been called 2-*acyclic* by the third author in [16, 19]; in that terminology, retractable groups even enjoy the stronger property of being 3-*acyclic*. For two cosets $gG[A]$ and $hG[B]$ in G condition (3.2) implies that their intersection is either empty or is a coset of the form $kG[A \cap B]$. We will often use a graph theoretic version of this fact: if, in the Cayley graph $\mathcal{G}$ of G, two vertices u and v are connected by an A-path as well as by a B-path then there is even an $(A \cap B)$-path $u \longrightarrow v$.

For $A \subseteq E$, an expansion $H \twoheadrightarrow G$ of E-groups is A-*stable* if the canonical morphism is injective when restricted to $H[A]$; it is k-*stable* (for $k < |E|$) if it is A-stable for every k-element subset A of E. For an E-group G and $A \subseteq E$ we denote the Cayley graph of $G[A]$, considered as an A-graph by $\mathcal{G}[A]$; this graph is weakly complete as an E-graph and we denote its trivial completion by $\overline{\mathcal{G}[A]}$. We arrive at our first basic construction. Here and in the following we use $\sqcup$ and $\bigsqcup$ to denote the disjoint union of graphs; recall the definition of the transition group of a complete graph (3.1).

Theorem 3.2. *Let $\mathcal{X}$ be a complete E-graph, $1 \leq k < |E|$ and suppose that the transition group $G = \mathscr{T}(\mathcal{X})$ is k-retractable. Then the transition group*

$$H := \mathscr{T}\Bigg(\mathcal{X} \sqcup \bigsqcup_{\substack{C \subseteq E \\ |C| = k}} \overline{\mathcal{G}[C]}\Bigg)$$

is $(k+1)$-retractable and is a k-stable expansion of G. Moreover, every k-stable expansion of H is also $(k+1)$-retractable.

Proof. We first show that H is a k-stable expansion of G. So, let $p \in \widetilde{E}^*$ be a word with $|\operatorname{co}(p)| \leq k$ und suppose that $[p]_G = 1$. We need to show that $[p]_H = 1$. In order to do so it is sufficient to show that, for every vertex v in $\mathcal{X} \sqcup \bigsqcup_{|C|=k} \overline{\mathcal{G}[C]}$ the path $\pi_v(p)$ which starts at v and has label p is a cycle. This is obvious for every $v \in \mathcal{X}$ and $v \in \overline{\mathcal{G}[A]}$ when A is a set of k letters for which $p \in \widetilde{A}^*$. So, let $B \subseteq E$ with $|B| = k$ and suppose that $p \notin \widetilde{B}^*$ which means that at least one element of the content of p does not belong to B and let v be a vertex of $\overline{\mathcal{G}[B]}$. Let p' be the word obtained from p by deletion of all letters from $\operatorname{co}(p) \setminus B$. Since G is k-retractable we have $[p']_G = 1$ and hence also $[p']_{G[B]} = 1$ since p' contains only letters from B. It follows that the path $\pi_v^{\mathcal{G}[B]}(p')$ is closed and hence so is $\pi_v^{\overline{\mathcal{G}[B]}}(p')$. Since the paths $\pi_v^{\overline{\mathcal{G}[B]}}(p)$ and $\pi_v^{\overline{\mathcal{G}[B]}}(p')$ meet exactly the same vertices — the two paths differ only in loop edges labelled by letters from $\operatorname{co}(p) \setminus B$ — it follows that $\pi_v^{\overline{\mathcal{G}[B]}}(p)$ is also closed. Altogether, $[p]_H = 1$ and the expansion $H \twoheadrightarrow G$ is k-stable.

Now let $A \subseteq E$ with $|A| = k+1$ and take a word p with content $\operatorname{co}(p) = A$ such that $[p]_H = 1$. In particular, $\pi_1^{\mathcal{H}}(p)$ is a closed path at vertex 1 in the Cayley graph $\mathcal{H}$ of H. Let $a \in A$ and $B = A \setminus \{a\}$. Then $|B| = k$ and $\overline{\mathcal{G}[B]}$ is a connected component of the complete graph that defines H. Hence there is a canonical graph morphism $\varphi\colon \mathcal{H} \twoheadrightarrow \overline{\mathcal{G}[B]}$ (for example φ_1, the one that maps $1 \to 1$). By this morphism the path $\pi_1^{\mathcal{H}}(p)$ is mapped to $\pi_1^{\overline{\mathcal{G}[B]}}(p)$ which therefore is also closed. In the latter path, every edge labelled a is a loop edge. Hence the path $\pi_1^{\overline{\mathcal{G}[B]}}(p_{a\to 1})$ is also closed, that is, $p_{a\to 1}$ labels a closed path (at 1) in $\mathcal{G}[B]$. It follows that $[p_{a\to 1}]_{G[B]} = 1$, hence $[p_{a\to 1}]_G = 1$ and thus $[p_{a\to 1}]_H = 1$ since H is a k-stable expansion of G. This applies to every letter occurring in p. That we may delete further letters from $p_{a\to 1}$ without changing the H-value now follows from the fact that H is a k-stable expansion of G and G is k-retractable. Moreover the arguments of this paragraph apply to every k-stable expansion of H thereby proving the assertion of the last sentence of the theorem. $\square$

The principal idea of the paper is to construct a series of E-generated permutation groups

$$G_1 \twoheadleftarrow G_2 \twoheadleftarrow \cdots \twoheadleftarrow G_{|E|} =: G \tag{3.3}$$

defined by an ascending sequence $\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \cdots \subseteq \mathcal{X}_{|E|}$ of complete E-graphs such that $G_k = \mathscr{T}(\mathcal{X}_k)$ is k-retractable and $G_{k+1} \twoheadrightarrow G_k$ is k-stable for every k. The crucial property of this sequence is the following:

> For every word $p \in \widetilde{E}^*$ on $k+1$ letters which forms a path $u \xrightarrow{p} v$ in $\mathcal{E}$ and every letter $a \in \operatorname{co}(p)$ either there is a

word q in the letters $\mathrm{co}(p) \setminus \{a\}$ such that $[p]_G = [q]_G$ and q also forms a path $u \xrightarrow{q} v$ in $\mathcal{E}$, or otherwise (if no such q exists) there is a component in $\mathcal{X}_{k+1} \setminus \mathcal{X}_k$ which guarantees that G_{k+1} avoids the relation $p = p_{a\to 1}$, so that a belongs to the content of $[p]_G$.

The graph theoretic constructions to be introduced in the following are designed to serve this purpose. In order to guarantee that $G_{k+1} \twoheadrightarrow G_k$ is k-stable, the components $\mathcal{L}$ of $\mathcal{X}_{k+1} \setminus \mathcal{X}_k$ will be assembled in a way that for every k-element subset A of E, every A-component of $\mathcal{L}$ appears as a subgraph of $\mathcal{X}_k$ or is isomorphic with $\mathcal{G}_k[A]$. This turns out to be a challenging task.

3.3. **Two crucial constructions: clusters and coset extensions.** We introduce two crucial constructions involving Cayley graphs. Let G be an E-generated group; for $A \subseteq E$ and $g \in G$, $g\mathcal{G}[A]$ has the obvious meaning: it denotes the A-component of the vertex g of $\mathcal{G}$ and is isomorphic (as an A-graph) with $\mathcal{G}[A]$ — we shall call such graphs *A-coset graphs* or simply *coset graphs* if the set of labels is understood. In the following subsections we shall construct new (bigger) graphs by gluing together disjoint copies of various coset graphs for different subsets $A \subseteq E$. In this context, the notation $v\mathcal{G}[A]$ where v is some vertex of a graph means that the A-component of v in the graph in question is isomorphic with the full A-coset graph $\mathcal{G}[A]$.

3.3.1. *Clusters.* Let G be an E-group, $A \subseteq E$ and assume that $G[A]$ is retractable. For every set $\mathbb{P}$ of proper subsets of A, the graph

$$\mathsf{CL}(G[A], \mathbb{P}) := \bigcup_{B \in \mathbb{P}} \mathcal{G}[B]$$

is an *A-cluster*. Note that $\mathsf{CL}(G[A], \mathbb{P})$ is the subgraph of $\mathcal{G}[A]$ which is spanned by all B-paths in $\mathcal{G}[A]$ starting at 1 where $B \in \mathbb{P}$. The *core* of the cluster is the subgraph formed by the intersection $\bigcap_{B\in\mathbb{P}} \mathcal{G}[B]$, and by retractability of $G[A]$, $\bigcap_{B\in\mathbb{P}} \mathcal{G}[B] = \mathcal{G}[\bigcap_{B\in\mathbb{P}} B]$. This core is always non-empty but may consist of the vertex 1 only; the subgraphs $\mathcal{G}[B]$, for $B \in \mathbb{P}$, are the *constituent cosets* of the cluster $\mathsf{CL}(G[A], \mathbb{P})$. Included in the definition of an A-cluster is for $\mathbb{P} = \{B\}$ every graph $\mathcal{G}[B]$ with $B \subsetneq A$. The structure of $\mathsf{CL}(G[A], \mathbb{P})$ as an A-graph actually only depends on the collection of the "small" subgroups $G[B]$, $B \in \mathbb{P}$ rather than on the entire group $G[A]$: indeed the cluster can be assembled from the constituents $\mathcal{G}[B]$ by forming their disjoint union and factoring by the congruence which identifies an element (vertex or edge) of some $\mathcal{G}[B]$ and some $\mathcal{G}[C]$ if and only if these elements coincide in $\mathcal{G}[B \cap C]$.

We first analyse the structure of B-components of A-clusters for $B \subsetneq A$. Let $\mathbb{P} = \{A_1, \ldots, A_k\}$ be a set of proper subsets of A and let $B \subsetneq A$; let $v \in G[A]$ and $v\mathcal{G}[B]$ be the B-component of v in $\mathcal{G}[A]$. For the intersection of $v\mathcal{G}[B]$ with the cluster we have

$$\mathsf{CL}(G[A],\mathbb{P}) \cap v\mathcal{G}[B] = \bigcup_{i=1}^{k} (\mathcal{G}[A_i] \cap v\mathcal{G}[B]).$$

The intersection $\mathcal{G}[A_i] \cap v\mathcal{G}[B]$ is either empty or a $(B \cap A_i)$-coset $v_i\mathcal{G}[B \cap A_i]$ for some (any) $v_i \in \mathcal{G}[A_i] \cap v\mathcal{G}[B]$. For our purpose we may assume that $\mathcal{G}[A_i] \cap v\mathcal{G}[B] \neq \varnothing$ for every i. Indeed, we may assume that we have already removed those sets A_i for which $\mathcal{G}[A_i] \cap v\mathcal{G}[B] = \varnothing$.

Lemma 3.3. *If $\mathcal{G}[A_i] \cap v\mathcal{G}[B] \neq \varnothing$ for $i = 1, \ldots, k$ then*

$$\mathcal{G}[A_1] \cap \cdots \cap \mathcal{G}[A_k] \cap v\mathcal{G}[B] \neq \varnothing.$$

Proof. Let $t \leq k$ and assume that we have already proved that

$$\bigcap_{i=1}^{t-1} \mathcal{G}[A_i] \cap v\mathcal{G}[B] \neq \varnothing.$$

So, let $u \in \bigcap_{i=1}^{t-1} \mathcal{G}[A_i] \cap v\mathcal{G}[B] = \mathcal{G}[\bigcap_{i=1}^{t-1} A_i] \cap v\mathcal{G}[B]$ and $w \in \mathcal{G}[A_t] \cap v\mathcal{G}[B]$. Let $p \in (\widetilde{A_1} \cap \cdots \cap \widetilde{A_{t-1}})^*$ be such that $[p]_G = u^{-1}$ and $q \in \widetilde{A_t}^*$ be such that

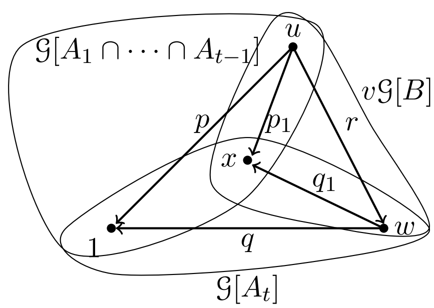

FIGURE 1

$[q]_G = w^{-1}$; moroever, let $r \in \widetilde{B}^*$ be a word which labels a path $u \longrightarrow w$ running entirely in $v\mathcal{G}[B]$ (recall that all this happens in $\mathcal{G}[A]$). Let p_1 and q_1 be, respectively, the words obtained from p and q by deletion of all letters not in B. Since $[pq^{-1}]_G = [r]_G$ and $r \in \widetilde{B}^*$ we have $[p_1q_1^{-1}]_G = [r]_G$, by retractability. Let $x := u \cdot p_1 = w \cdot q_1$. Then $p^{-1}p_1$ labels a path $1 \longrightarrow x$ and so does $q^{-1}q_1$. Since $p^{-1}p_1 \in (\widetilde{A_1} \cap \cdots \cap \widetilde{A_{t-1}})^*$ and $q^{-1}q_1 \in \widetilde{A_t}^*$ it follows that $x \in \mathcal{G}[A_1 \cap \cdots \cap A_{t-1}] \cap \mathcal{G}[A_t] = \mathcal{G}[A_1 \cap \cdots \cap A_t]$. From $x = u \cdot p_1$ and $p_1 \in \widetilde{B}^*$ it follows that $x \in u\mathcal{G}[B] = v\mathcal{G}[B]$, altogether $x \in \mathcal{G}[A_1 \cap \cdots \cap A_t] \cap v\mathcal{G}[B]$. $\square$

If we consider the automorphism of $\mathcal{G}$ induced by left multiplication by x^{-1} (x as in the proof above) then we see that the intersection

$$\mathsf{CL}(G[A],\mathbb{P}) \cap x\mathcal{G}[B] = \bigcup_{i=1}^{k} (\mathcal{G}[A_i] \cap x\mathcal{G}[B])$$

is isomorphic with the B-cluster $\mathsf{CL}(G[B], \mathbb{O})$ where $\mathbb{O} = \{B \cap A_i \mid A_i \in \mathbb{P}\}$ (some of the sets $B \cap A_i$ may be empty) which perhaps degenerates to a full B-coset. This allows us to characterise the B-components of A-clusters for $B \subsetneq A$.

Corollary 3.4. *Let $\mathbb{P}$ be a set of proper subsets of A and $B \subsetneq A$. Then every B-component of the cluster $\mathsf{CL}(G[A], \mathbb{P})$ is either a B-coset, that is, isomorphic with $\mathcal{G}[B]$, or isomorphic with the B-cluster $\mathsf{CL}(G[B], \mathbb{O})$ where $\mathbb{O} = \{C \cap B \mid C \in \mathbb{P}\}$ (note that some $C \cap B$ may be empty).*

Proof. The intersection $\mathsf{CL}(G[A], \mathbb{P}) \cap v\mathcal{G}[B]$ is either the B-coset $v\mathcal{G}[B]$ itself (if it is contained in some constituent $\mathcal{G}[C]$ with $C \in \mathbb{P}$) or otherwise is isomorphic with the B-cluster $\mathsf{CL}(G[B], \mathbb{O})$, as indicated above. Now let v be a vertex of $\mathsf{CL}(G[A], \mathbb{P})$; then the B-component $\mathcal{B}$ of v in $\mathsf{CL}(G[A], \mathbb{P})$ is certainly contained in $\mathsf{CL}(G[A], \mathbb{P}) \cap v\mathcal{G}[B]$. Since the latter intersection is a B-cluster it is connected and therefore $\mathcal{B}$ must coincide with this intersection. ☐

Corollary 3.5. *Let $B, C \subsetneq A$; then the intersection $\mathcal{B} \cap \mathcal{C}$ of a B-component $\mathcal{B}$ with a C-component $\mathcal{C}$ of an A-cluster CL is either empty or a $B \cap C$-coset or a $(B \cap C)$-cluster.*

Proof. As mentioned above, $\mathcal{B} = \mathsf{CL} \cap v\mathcal{G}[B]$ and $\mathcal{C} = \mathsf{CL} \cap w\mathcal{G}[C]$ for some cosets $v\mathcal{G}[B]$ and $w\mathcal{G}[C]$. The latter two have either empty intersection or their intersection is a $(B \cap C)$-coset $u\mathcal{G}[B \cap C]$ from which the claim follows. ☐

We will need a generalisation of clusters which we are going to present next. Let again be $G[A]$ be retractable, $\mathbb{P}$ be a set of proper subsets of A, v be a vertex of $\mathsf{CL}(G[A], \mathbb{P})$ and $B \subsetneq A$. Under these assumptions we define

$$\mathsf{CL}(G[A], \mathbb{P}) ⓥ \mathcal{G}[B] := \bigcup_{C \in \mathbb{P}} \mathcal{G}[C] \cup v\mathcal{G}[B]$$

considered as a subgraph of $\mathcal{G}[A]$ and call the latter graph a *B-augmented A-cluster* or, more specifically, the *B-augmentation of* $\mathsf{CL}(G[A], \mathbb{P})$ *at* v. We have already seen that the intersection $\mathsf{CL}(G[A], \mathbb{P}) \cap v\mathcal{G}[B]$ is a B-component of $\mathsf{CL}(G[A], \mathbb{P})$. It follows that the structure of the graph $\mathsf{CL}(G[A], \mathbb{P}) ⓥ \mathcal{G}[B]$ only depends on the collection $\{G[C] \mid C \in \mathbb{P}\}$, the vertex v and $G[B]$ rather than on the entire group $G[A]$. Indeed, as mentioned earlier, the structure of $\mathsf{CL}(G[A], \mathbb{P})$ depends only on the graphs $\mathcal{G}[C]$ for $C \in \mathbb{P}$ and the B-component of v is a certain B-cluster $\mathcal{B}$ which is isomorphic with a certain subgraph of $\mathcal{G}[B]$ via a monomorphism $\iota \colon \mathcal{B} \to \mathcal{G}[B]$. The augmented cluster $\mathsf{CL}(G[A], \mathbb{P}) ⓥ \mathcal{G}[B]$ then can be obtained as the disjoint union of $\mathsf{CL}(G[A], \mathbb{P})$ and $\mathcal{G}[B]$ factored by the congruence whose non-singleton classes are $\{x, \iota(x)\}$ for all $x \in \mathcal{B}$ (x an edge or a vertex).

As the last result in this subsection we need to clarify, for $C \subsetneq A$, the structure of C-components of B-augmented A-clusters. These will turn out to be $(B \cap C)$-augmented C-clusters. We noticed already that every C-component of an A-cluster is a C-cluster (or a C-coset).

Corollary 3.6. *Let $B, C \subsetneq A$ and let $G[A]$ be retractable; then every C-component of a B-augmented A-cluster is a $(B \cap C)$-augmented C-cluster (which includes C-clusters as a special case).*

Proof. Let the group G and $A, B, C \subseteq E$ be as in the statement of the corollary. Let $\mathsf{CL}(G[A], \mathbb{P})ⓥ\mathcal{G}[B]$ be a B-augmentation of the A-cluster $\mathsf{CL}(G[A], \mathbb{P})$ and let u be a vertex of this cluster. If the C-component $\mathcal{C}$ of u in $\mathsf{CL}(G[A], \mathbb{P})$ has empty intersection with the B-component $\mathcal{B}$ of v in $\mathsf{CL}(G[A], \mathbb{P})$ then $\mathcal{C}$ coincides with the C-component of u in the augmented cluster and we are done as this is just a C-cluster (or a C-coset). Now assume that $\mathcal{C} \cap \mathcal{B} \neq \varnothing$ with w a vertex in $\mathcal{C} \cap \mathcal{B}$. We know that $\mathcal{C} \cap \mathcal{B}$ is a $(C \cap B)$-cluster (Corollary 3.5) or a $(C \cap B)$-coset and the C-component of w within $v\mathcal{G}[B] = w\mathcal{G}[B]$ consists exactly of the coset $w\mathcal{G}[B \cap C]$. It follows that the C-component of w in $\mathsf{CL}(G[A], \mathbb{P})ⓥ\mathcal{G}[B]$ coincides with $\mathcal{C} \cup w\mathcal{G}[B \cap C] = \mathcal{C}ⓦ\mathcal{G}[B \cap C]$ which is a $(B \cap C)$-augmentation of the C-cluster $\mathcal{C}$. $\square$

3.3.2. *Coset extensions.* The second construction can be seen as a generalisation of clusters but is more involved. Let us fix an E-group G and a set $A \subseteq E$ of size $|A| \geq 2$ and assume that G is A-retractable. Let $\mathcal{K}$ be a connected A-subgraph of the Cayley graph $\mathcal{G}$ of G. We recall that being an A-subgraph means that all labels of edges of $\mathcal{K}$ belong to $\widetilde{A}$ (but not necessarily all such letters actually need to occur in $\mathcal{K}$). For some set $B \subsetneq A$ let $\mathcal{B} = v\mathcal{K}[B]$ be some B-component of $\mathcal{K}$; this graph is embedded in $v\mathcal{G}[B] \cong \mathcal{G}[B]$. Moreover, for $B_1, B_2 \subsetneq B$ any B_1- and B_2-components $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ are also embedded in $v\mathcal{G}[B]$.

Definition 3.7. The graph $\mathcal{K}$ is *admissible for $^{\subsetneq}A$-coset extension* if, for all choices $B_1, B_2 \subsetneq B \subsetneq A$ and all possible components $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B} \subseteq v\mathcal{G}[B]$ (in the situation described above) the following holds

$$\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing \text{ in } \mathcal{B} \Longrightarrow v_1\mathcal{G}[B_1] \cap v_2\mathcal{G}[B_2] = \varnothing \text{ in } v\mathcal{G}[B] \tag{3.4}$$

for some (equivalently all) vertices $v_1 \in \mathcal{B}_1, v_2 \in \mathcal{B}_2$.

In other words, the patterns depicted in Figure 2 are forbidden in the context of a graph $\mathcal{K}$ that is admissible for $^{\subsetneq}A$-coset extension (the second picture is for the case $B_1 = B_2$). In [19] the condition formulated in Definition 3.7 has been termed the *freeness* condition of the embedded graph $\mathcal{K}$. We note that, if $\mathcal{K}$ is admissible for $^{\subsetneq}A$-coset extension, then, for every $B \subseteq A$, every B-component $v\mathcal{K}[B]$ is admissible for $^{\subsetneq}B$-coset extension. Now let $\mathcal{K}$ be a subgraph of $\mathcal{G}$ that is admissible for $^{\subsetneq}A$-coset extension and fix a set $B \subsetneq A$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be all the B-components of $\mathcal{K}$. For every $i = 1, \ldots, k$ select a vertex $v_i \in \mathcal{B}_i$. Then, in $\mathcal{G}$, the coset $v_i\mathcal{G}[B]$ contains $\mathcal{B}_i$ as a subgraph. Let now $\mathsf{CE}(G, \mathcal{K}; B)$ be the graph obtained by extending each component $\mathcal{B}_i$ in $\mathcal{K}$ to the entire coset $v_i\mathcal{G}[B]$. In other

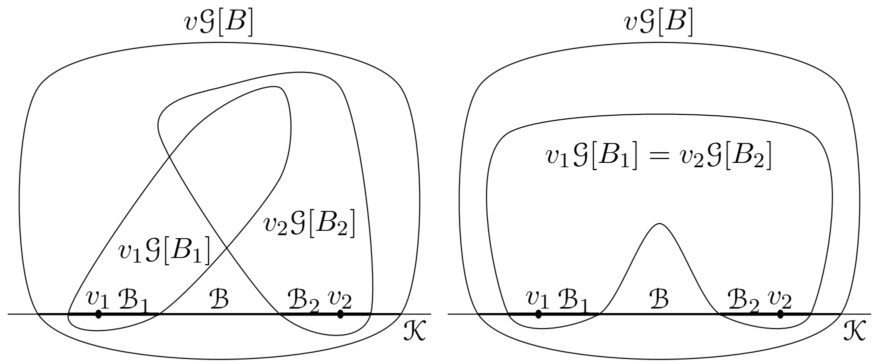

FIGURE 2

words, $\mathsf{CE}(G, \mathcal{K}; B)$ is the graph obtained by attaching in $\mathcal{K}$ to each vertex v_i a copy $v_i\mathcal{G}[B]$ of $\mathcal{G}[B]$ and then identifying all of $\mathcal{B}_i$ with its copy inside $v_i\mathcal{G}[B]$, but without performing any further identification (of vertices and/or edges). The graph $\mathsf{CE}(G, \mathcal{K}; B)$ thus appears as a bunch of pairwise disjoint copies of $\mathcal{G}[B]$, connected by edges labelled by letters from $A \setminus B$. The union of the latter edges with all $\mathcal{B}_i$ then spans the graph $\mathcal{K}$.

We give a more formal definition of $\mathsf{CE}(G, \mathcal{K}; B)$. Let $\mathcal{K}$ be given with B-components $\mathcal{B}_1, \ldots, \mathcal{B}_k$ and selected vertices $v_i \in \mathcal{B}_i$ for $i = 1, \ldots, k$. For every i let $\iota_i \colon \mathcal{B}_i \to \mathcal{G}[B]$ be the unique graph monomorphism mapping v_i to 1. Then

$$\mathsf{CE}(G, \mathcal{K}; B) := \mathcal{K} \cup \bigcup_{i=1}^{k} \mathcal{G}[B] \times \{i\} \,/\Theta \tag{3.5}$$

where Θ is the equivalence relation all of whose non-singleton equivalence classes are exactly the two element sets

$$\{x, (\iota_i(x), i)\} \text{ with } x \in \mathcal{B}_i,\ i = 1, \ldots, k$$

where x denotes a vertex or and edge of $\mathcal{B}_i$. The union on the right hand side of (3.5) is a union of pairwise disjoint connected graphs and Θ is certainly a congruence relation. The resulting graph $\mathsf{CE}(G, \mathcal{K}; B)$ is the *B-coset extension of the A-graph* $\mathcal{K}$; in this context, the subgraph $\mathcal{K}$ of $\mathsf{CE}(G, \mathcal{K}; B)$ is the *skeleton* of $\mathsf{CE}(G, \mathcal{K}; B)$ and the subgraphs $v_i\mathcal{G}[B]$ are the *constituent cosets* of $\mathsf{CE}(G, \mathcal{K}; B)$. Definition 3.7 implies for $C \subsetneq B \subsetneq A$ (by taking $B_1 = C, B_2 = \varnothing$ or $B_1 = C = B_2$) that the identity morphism $\mathcal{K} \to \mathcal{K}$ extends to an embedding $\mathsf{CE}(G, \mathcal{K}, C) \hookrightarrow \mathsf{CE}(G, \mathcal{K}; B)$ hence $\mathsf{CE}(G, \mathcal{K}; C)$ can be seen as a subgraph of $\mathsf{CE}(G, \mathcal{K}; B)$ in this case. So, the set of all graphs $\mathsf{CE}(G, \mathcal{K}; B)$ for $B \subsetneq A$ is partially ordered by inclusion, and for $B, C \subsetneq A$ we have

$$\mathsf{CE}(G, \mathcal{K}; B) \cap \mathsf{CE}(G, \mathcal{K}; C) = \mathsf{CE}(G, \mathcal{K}; B \cap C). \tag{3.6}$$

Now let $\mathbb{P}$ be a set of proper subsets of A. Then the $\mathbb{P}$-*coset extension of* $\mathcal{K}$ is defined as

$$\mathsf{CE}(G,\mathcal{K};\mathbb{P}) := \bigcup_{B\in\mathbb{P}} \mathsf{CE}(G,\mathcal{K};B)\times\{B\} \,\big/\, \Psi \tag{3.7}$$

where Ψ is the congruence defined on the disjoint union of all B-coset extensions $\mathsf{CE}(G,\mathcal{K};B)$ with $B\in\mathbb{P}$, by setting

$$(x_1,B_1)\ \Psi\ (x_2,B_2) :\Longleftrightarrow x_1 = x_2 \in \mathsf{CE}(G,\mathcal{K};B_1\cap B_2).$$

In other words, an edge or a vertex of $\mathsf{CE}(G,\mathcal{K};B_1)$ is identified with one in $\mathsf{CE}(G,\mathcal{K};B_2)$ if they represent the same element in $\mathsf{CE}(G,\mathcal{K};B_1\cap B_2)$. Provided that $B\in\mathbb{P}$, the coset extension $\mathsf{CE}(G,\mathcal{K};B)$ is embedded in $\mathsf{CE}(G,\mathcal{K};\mathbb{P})$ via $x\mapsto(x,B)\Psi$ where $(x,B)\Psi$ denotes the Ψ-class of (x,B).

Geometrically, the coset extension $\mathsf{CE}(G,\mathcal{K};\mathbb{P})$ can be viewed as follows. For every $B\in\mathbb{P}$ consider $\mathsf{CE}(G,\mathcal{K};B)$ and attach these graphs to each other by identification of their skeleton $\mathcal{K}$, then form the largest E-graph quotient (that is, perform all identifications necessary to obtain an E-graph, but not more). If every label of $\mathcal{K}$ appears in some member B of $\mathbb{P}$, then $\mathsf{CE}(G,\mathcal{K},\mathbb{P})$ is weakly complete since every edge of $\mathsf{CE}(G,\mathcal{K};\mathbb{P})$ occurs in some coset subgraph $v\mathcal{G}[B]$. Most relevant will be the case $\mathbb{P}=\mathbb{P}_A$, the set of *all* proper subsets of A: we call $\mathsf{CE}(G,\mathcal{K};\mathbb{P}_A)$ the *full* ${}^{\subsetneq}A$-coset extension of $\mathcal{K}$. In case $\mathcal{K}=\{v\}$ (one vertex, no edge) the $\mathbb{P}$-coset extension $\mathsf{CE}(G,\mathcal{K};\mathbb{P})$ reduces to the cluster $\mathsf{CL}(G[A],\mathbb{P})$.

Remark 3.8. An A-graph $\mathcal{K}$ which is admissible for ${}^{\subsetneq}A$-coset extension may actually only contain edges labelled by letters from some set $B\subsetneq A$. In this case $\mathsf{CE}(G,\mathcal{K};B)\cong\mathcal{G}[B]$; however, this is not in conflict with the definition of the full ${}^{\subsetneq}A$-coset extension. For sets $C\subsetneq A$ with $C\nsubseteq B$, the C-components of $\mathcal{K}$ coincide with the $C\cap B$-components, but nevertheless every such $C\cap B$-component is extended to a full C-coset $v\mathcal{G}[C]$ in order to get $\mathsf{CE}(G,\mathcal{K};C)$.

We continue with further investigations of ${}^{\subsetneq}A$-coset extensions.

Proposition 3.9. *Let $\mathcal{K}\subseteq\mathcal{G}[A]$ be admissible for ${}^{\subsetneq}A$-coset extension and $\mathbb{P}$ be a set of proper subsets of A. Then the inclusion monomorphism $\iota\colon\mathcal{K}\hookrightarrow\mathcal{G}[A]$ admits a unique extension to a graph morphism $\iota_{\mathbb{P}}\colon\mathsf{CE}(G,\mathcal{K};\mathbb{P})\to\mathcal{G}[A]$.*

Proof. We first establish a unique extension $\iota_B\colon\mathsf{CE}(G,\mathcal{K};B)\to\mathcal{G}[A]$. Let $\mathcal{B}_1,\dots,\mathcal{B}_k$ be all B-components of $\mathcal{K}$ with selected vertices $v_i\in\mathcal{B}_i$ for all i. Then for every i there is a unique graph monomorphism $\kappa_i\colon\mathcal{G}[B]\times\{i\}\to\mathcal{G}[A]$ such that $\kappa_i(1,i)=v_i$. The image of κ_i coincides with the coset subgraph $v_i\mathcal{G}[B]$ of $\mathcal{G}[A]$. Then, the union $\kappa:=\iota\cup\bigcup_{i=1}^k\kappa_i$ is a morphism

$$\kappa\colon\mathcal{K}\cup\bigcup_{i=1}^{k}\mathcal{G}[B]\times\{i\}\to\mathcal{G}[A]$$

for which, for all i and $x \in \mathcal{B}_i$,

$$\kappa(x) = \iota(x) = x = \kappa_i(\iota_i(x), i) = \kappa(\iota_i(x), i)$$

where $\iota_i \colon \mathcal{B}_i \to \mathcal{G}[B]$ is the unique graph monomorphism mapping v_i to 1 that occurred in the definition of $\mathsf{CE}(G, \mathcal{K}; B)$. It follows that the congruence Θ in (3.5) is contained in the *kernel* of κ (that is, the equivalence relation induced by κ on its domain) and hence κ factors through $\mathsf{CE}(G, \mathcal{K}; B)$ as $\kappa = \iota_B \circ \pi_\Theta$ (where π_Θ is the canonical projection $\pi_\Theta(x) = x\Theta$). The morphism ι_B is not injective in general.

Next consider the disjoint union

$$\bigcup_{B \in \mathbb{P}} \mathsf{CE}(G, \mathcal{K}; B) \times \{B\}$$

and let

$$\kappa_{\mathbb{P}} := \bigcup_{B \in \mathbb{P}} \bar{\iota_B} \colon \bigcup_{B \in \mathbb{P}} \mathsf{CE}(G, \mathcal{K}; B) \times \{B\} \to \mathcal{G}[A]$$

where $\bar{\iota_B} \colon \mathsf{CE}(G, \mathcal{K}, B) \times \{B\} \to \mathcal{G}[A]$ is defined by $\bar{\iota_B}(x, B) = \iota_B(x)$. Similar to Θ and κ, the congruence Ψ that occurred in (3.7) is contained in the kernel of $\kappa_{\mathbb{P}}$, hence $\kappa_{\mathbb{P}}$ factors through $\mathsf{CE}(G, \mathcal{K}; \mathbb{P})$ as $\kappa_{\mathbb{P}} = \iota_{\mathbb{P}} \circ \pi_\Psi$ for some unique morphism $\iota_{\mathbb{P}} \colon \mathsf{CE}(G, \mathcal{K}; \mathbb{P}) \to \mathcal{G}[A]$ (with π_Ψ being again the projection $x \mapsto x\Psi$). $\square$

We note that $\iota_{\mathbb{P}}$ is injective when restricted either to the skeleton $\mathcal{K}$ or to any constituent coset $v\mathcal{G}[B]$ for $B \subsetneq A$. We are able to sharpen an earlier remark. Let $\mathcal{K}$ be a connected A-graph admissible for ${}^{\subsetneq}A$-coset extension, let $B \subsetneq A$ and let $\mathcal{B} \subseteq \mathcal{K}$ be a B-component of $\mathcal{K}$ with v a vertex of $\mathcal{B}$. By construction of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$,

$$v \in \mathcal{B} \subseteq v\mathcal{G}[B] \subseteq \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A).$$

But $\mathcal{B}$ is itself admissible for ${}^{\subsetneq}B$-coset extension and hence $\mathsf{CE}(G, \mathcal{B}; \mathbb{P}_B)$ is well defined. Admissibility of $\mathcal{K}$ (Definition 3.7) implies that in this case the morphism $\iota_{\mathbb{P}_B} \colon \mathsf{CE}(G, \mathcal{B}; \mathbb{P}_B) \to \mathcal{G}[B]$ of Proposition 3.9 is injective, so that we do have, in fact:

Lemma 3.10. *Let G be A-retractable and $\mathcal{K}$ be a subgraph of $\mathcal{G}[A]$ which is admissible for ${}^{\subsetneq}A$-coset extension. Let $B \subsetneq A$ with $|B| \geq 2$; then every B-component $\mathcal{B}$ of $\mathcal{K}$ is admissible for ${}^{\subsetneq}B$-coset extension and the morphism $\iota_{\mathbb{P}_B} \colon \mathsf{CE}(G, \mathcal{B}; \mathbb{P}_B) \to \mathcal{G}[B]$ is injective. In particular, for any vertex $v \in \mathcal{B}$,*

$$v \in \mathcal{B} \subseteq \mathsf{CE}(G, \mathcal{B}; \mathbb{P}_B) \subseteq v\mathcal{G}[B] \subseteq \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A).$$

Another consequence is the following. In terms of [19] this means that a graph which is admissible for ${}^{\subsetneq}A$-coset extension is 2-acyclic.

Lemma 3.11. *Suppose that G is A-retractable and that the graph $\mathcal{K}$ is admissible for ${}^{\subsetneq}A$-coset extension. Then, for any $B, C \subsetneq A$, $B \neq C$, the intersection $\mathcal{B} \cap \mathcal{C}$ of every B-component $\mathcal{B}$ and every C-component $\mathcal{C}$ of $\mathcal{K}$ is connected and hence is a $(B \cap C)$-component.*

Proof. Suppose that $B \neq C$ and let u, v be vertices of $\mathcal{B} \cap \mathcal{C}$ and assume that they belong to distinct components of $\mathcal{B} \cap \mathcal{C}$. Admissibility of $\mathcal{K}$ (by taking $B_1 = B \cap C = B_2$) implies that the cosets $u\mathcal{G}[B \cap C]$ and $v\mathcal{G}[B \cap C]$ are disjoint (that is, distinct), and both cosets are contained in $u\mathcal{G}[B] = v\mathcal{G}[B]$ as well as $u\mathcal{G}[C] = v\mathcal{G}[C]$. Consider the graph morphism $\iota_{\mathbb{P}_A} \colon \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A) \to \mathcal{G}[A]$. It maps the cosets $u\mathcal{G}[B]$ as well as $v\mathcal{G}[C]$ injectively to the corresponding coset subgraphs of $\mathcal{G}[A]$. Since $u\mathcal{G}[B \cap C]$ and $v\mathcal{G}[B \cap C]$ are disjoint it follows that the intersection of the cosets $u\mathcal{G}[B]$ and $v\mathcal{G}[C]$ (in $\mathcal{G}[A]$) is disconnected as it has at least the two components $u\mathcal{G}[B \cap C]$ and $v\mathcal{G}[B \cap C]$. However, the latter contradicts the assumption that $G[A]$ is retractable. □

3.3.3. *Augmented coset extensions.* Similarly to clusters we require augmented coset extensions. Again fix an E-group G, let $A \subseteq E$ with $|A| \geq 2$ and assume that $G[A]$ is retractable. Let $\mathcal{K} \subseteq \mathcal{G}[A]$ be admissible for ${}^{\subsetneq}A$-coset extension. Recall that the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ can be seen as the union $\bigcup_{B \subsetneq A} \mathsf{CE}(G, \mathcal{K}, B)$ where for $B, C \subsetneq A$, $\mathsf{CE}(G, \mathcal{K}; B) \subseteq \mathsf{CE}(G, \mathcal{K}; C)$ if and only if $B \subseteq C$ and also (3.6) holds. Every vertex x of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ is sitting in some $\mathsf{CE}(G, \mathcal{K}; B)$, and, inside $\mathsf{CE}(G, \mathcal{K}; B)$ in a unique *constituent coset* $v\mathcal{G}[B]$ with $v \in \mathcal{K}$. The vertex v is not unique, but unique is its B-component $v\mathcal{K}[B]$. In this situation we say that the pair (B, v) *supports* the vertex x or *provides support* for the vertex x in $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$; the *size* of this support is $|B|$. This actually means that the skeleton $\mathcal{K}$ may be accessed from the vertex x by a B-path whose terminal vertex is v. We say that (B, v) provides *unique minimal support* if, whenever (C, w) provides support for x then $B \subseteq C$ and $v\mathcal{K}[B] \subseteq w\mathcal{K}[C]$. Now let $\mathcal{J}$ be a subgraph of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$; for a set $B \subsetneq A$ and a vertex $v \in \mathcal{K}$ we say that (B, v) provides *unique minimal support for* $\mathcal{J}$ if (B, v) supports some vertex x of $\mathcal{J}$, and if some pair (C, w) supports some vertex y of $\mathcal{J}$ then $B \subseteq C$ and $v\mathcal{K}[B] \subseteq w\mathcal{K}[C]$. In this case we say that the unique minimal support of $\mathcal{J}$ is *attained* at the vertex x. Notice that the condition $v\mathcal{K}[B] \subseteq w\mathcal{K}[C]$ implies that for the involved constituent cosets the inclusion $v\mathcal{G}[B] \subseteq w\mathcal{G}[C] = v\mathcal{G}[C]$ holds. We come to a crucial property the full ${}^{\subsetneq}A$-coset extension of a graph $\mathcal{K}$ may or may not have.

Definition 3.12. The full coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ has the *cluster property* if, for every $B \subsetneq A$ the following hold:

1. every B-component $\mathcal{B}$ of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ which has empty intersection with the skeleton $\mathcal{K}$ is a B-cluster or a full B-coset;
2. every $\mathcal{B}$ of (1) has unique minimal support which is attained at some vertex x of the core of $\mathcal{B}$ (if $\mathcal{B}$ is a cluster).

Note that minimal support will typically *not* be attained at all core vertices. We first show that the cluster property implies that components of the coset extension intersect nicely, that is, the coset extension is 2-acyclic in terms of [19].

Proposition 3.13. *Suppose that G is A-retractable, that $\mathcal{K} \subseteq \mathcal{G}[A]$ is admissible for ${}^{\subsetneq}\!A$-coset extension and that the full ${}^{\subsetneq}\!A$-coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ has the cluster property. Then, for all pairs $B, C \subsetneq A$ the intersection $\mathcal{B} \cap \mathcal{C}$ of any B-component $\mathcal{B}$ and any C-component $\mathcal{C}$ is connected and hence is a $(B \cap C)$-component of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$.*

Proof. We consider several cases and start with the most difficult one: suppose that both $\mathcal{B}$ and $\mathcal{C}$ have empty intersection with the skeleton $\mathcal{K}$. We need to show that $\mathcal{B}\cap\mathcal{C}$ is connected. We know that $\mathcal{B}$ is a B-cluster, $\mathcal{C}$ is a C-cluster, that is, $\mathcal{B} \cong \mathsf{CL}(G[B], \{B_1, \dots, B_k\})$ and $\mathcal{C} \cong \mathsf{CL}(G[C], \{C_1, \dots, C_l\})$ for $B_i \subsetneq B$ and $C_j \subsetneq C$; it may also happen that $k = 1$ and/or $l = 1$ in which case it may happen that $B_1 = B$ and/or $C_1 = C$ (that is, $\mathcal{B}$ and/or $\mathcal{C}$ is a B-coset and/or C-coset) — the argument for this subcase is similar but more simple. Take a vertex x in the core of $\mathcal{B}$, y a vertex in the core of $\mathcal{C}$ and let the unique minimal support of $\mathcal{B}$ be (M, m) (attained at x) and the unique minimal support of $\mathcal{C}$ be (N, n) (attained at y). Then $\mathcal{B} = \bigcup_{i=1}^{k} x\mathcal{G}[B_i]$ and $\mathcal{C} = \bigcup_{j=1}^{l} y\mathcal{G}[C_j]$. Let $u_1 \neq u_2$ be vertices of $\mathcal{B} \cap \mathcal{C}$; we may assume that $u_1 \in x\mathcal{G}[B_1] \cap y\mathcal{G}[C_1]$ and $u_2 \in x\mathcal{G}[B_2] \cap y\mathcal{G}[C_2]$. Recall that $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ has the cluster property. The vertices u_1 and u_2 (as $\varnothing$-components $\{u_1\}$ and $\{u_2\}$) also have unique minimal support (F_1, v_1) and (F_2, v_2), say. Then $M, N \subseteq F_1, F_2$ and even more holds, namely

$$\begin{aligned} & m\mathcal{G}[M], n\mathcal{G}[N] \subseteq m\mathcal{G}[F_1] = v_1\mathcal{G}[F_1] = n\mathcal{G}[F_1] \\ \text{and} \quad & m\mathcal{G}[M], n\mathcal{G}[N] \subseteq m\mathcal{G}[F_2] = v_2\mathcal{G}[F_2] = n\mathcal{G}[F_1]. \end{aligned}$$

The equality $m\mathcal{G}[F_1] = v_1\mathcal{G}[F_1]$ follows from the fact that (F_1, v_1) provides some support for $\mathcal{B}$ while (M, m) provides unique minimal support for $\mathcal{B}$ hence $M \subseteq F_1$ and $m \in m\mathcal{G}[M] \subseteq v_1\mathcal{G}[F_1]$; likewise, (F_1, v_1) provides some support for $\mathcal{C}$ while (N, n) provides unique minimal support for $\mathcal{C}$, hence $N \subseteq F_1$ and $n \in n\mathcal{G}[N] \subseteq v_1\mathcal{G}[F_1]$ which implies $v_1\mathcal{G}[F_1] = n\mathcal{G}[F_1]$. The remaining two equalities are proved in the same fashion.

Consequently, $v_1\mathcal{G}[F_1] \cap v_2\mathcal{G}[F_2] \neq \varnothing$ and hence also $v_1\mathcal{K}[F_1] \cap v_2\mathcal{K}[F_2] \neq \varnothing$. By Lemma 3.11, this intersection is an F-component of $\mathcal{K}$ for $F = F_1 \cap F_2$. It follows that the subgraph of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ formed by the union $v_1\mathcal{G}[F_1] \cup v_2\mathcal{G}[F_2]$ is isomorphic with the cluster $\mathsf{CL}(G[A], \{F_1, F_2\})$. Setting $B' := (B_1 \cap F_1) \cup (B_2 \cap F_2)$ and $C' := (C_1 \cap F_1) \cup (C_2 \cap F_2)$ we see that u_1 and u_2 belong to the same B'- as well as C'-component of that cluster, the intersection of which is a $(B' \cap C')$-component of that cluster. Consequently, u_1 and u_2 are in the same $(B' \cap C')$-component of $\mathsf{CL}(G[A]; \{F_1, F_2\})$ and hence in the same $(B \cap C)$-component of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$, see Figure 3.

Next consider the case when $\mathcal{B}$ has non-empty intersection with the skeleton $\mathcal{K}$. In this case, $\mathcal{B} = v\mathcal{G}[B]$ for some vertex $v \in \mathcal{K}$. Let $u_1 \in \mathcal{B}\cap\mathcal{C}_1$, $u_2 \in \mathcal{B}\cap\mathcal{C}_2$, and let x be a vertex in the core of $\mathcal{C}$ which attains minimal support of $\mathcal{C}$. In this case, (B, v) supports u_1 as well as u_2 and therefore also x, so that $u_1, x, u_2 \in \mathcal{B} = v\mathcal{G}[B]$, see Figure 4. Hence there is a $(B \cap C)$-path $u_1 \longrightarrow x$ and also a $(B\cap C)$-path $x \longrightarrow u_2$, altogether there is a $(B\cap C)$-path

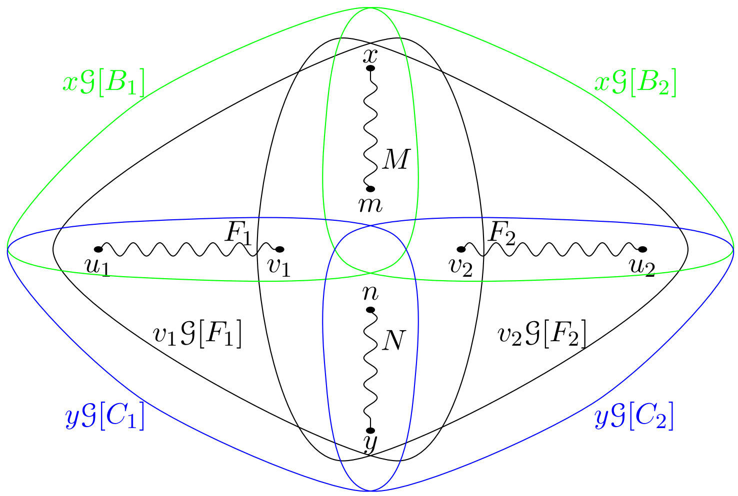

FIGURE 3

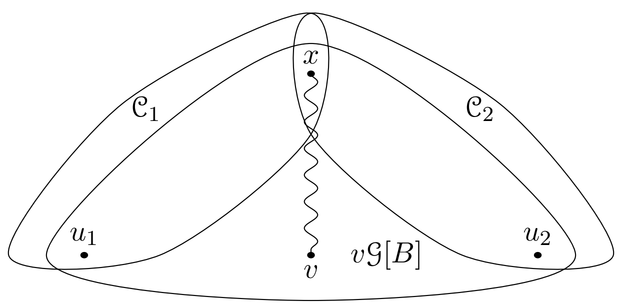

FIGURE 4

$u_1 \longrightarrow u_2$. Finally, the case when $\mathcal{B}$ as well as $\mathcal{C}$ have non-empty intersection with the skeleton $\mathcal{K}$ is obvious, since in this case $\mathcal{B}\cap\mathcal{C}$ is a $(B\cap C)$-coset. $\square$

We are led to a further construction. Let $\mathcal{K}$ be admissible for ${}^{\subsetneq}A$-coset extension and suppose that the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(G,\mathcal{K},\mathbb{P}_A)$ has the cluster property. For a vertex $v \in \mathsf{CE}(G,\mathcal{K};\mathbb{P}_A)$ and some $B \subsetneq A$ the B-component $\mathcal{B}$ of v is either a B-coset $v\mathcal{G}[B]$ (in this case, $\mathcal{B}$ may or may not intersect with the skeleton $\mathcal{K}$) or a proper B-cluster (in which case it does not intersect with the skeleton $\mathcal{K}$). In any case, $\mathcal{B}$ embeds into $\mathcal{G}[B]$ via some graph monomorphism $\iota\colon \mathcal{B} \hookrightarrow \mathcal{G}[B]$ (which is unique if one additionally assumes that $\iota(v) = 1$). We define the *B-augmentation at v of* $\mathsf{CE}(G,\mathcal{K};\mathbb{P}_A)$ by

$$\mathsf{CE}(G,\mathcal{K},\mathbb{P}_A) ⓥ \mathcal{G}[B] := \mathsf{CE}(G,\mathcal{K};\mathbb{P}_A) \sqcup \mathcal{G}[B] \,/\Omega$$

where Ω is the congruence the set of whose non-singleton congruence classes is given by $\{\{x,\iota(x)\} \mid x \in \mathcal{B}\}$. We note that $\mathsf{CE}(G,\mathcal{K};\mathbb{P}_A)ⓥ\mathcal{G}[B]$ can be written as the union

$$\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A) \cup v\mathcal{G}[B]$$

of the two subgraphs $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ and $v\mathcal{G}[B]$ whose intersection is just the B-component $\mathcal{B}$ of v in $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$.

Proposition 3.14. *Let $B, C \subsetneq A$ and $\mathcal{K}$ be admissible for ${}^{\subsetneq}A$-coset extension such that the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ enjoys the cluster property. Then every C-component of some B-augmented full coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)ⓥ\mathcal{G}[B]$ is either a C-coset or a $(B \cap C)$-augmented C-cluster.*

Proof. Let $\mathcal{C}$ be a C-component of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)ⓥ\mathcal{G}[B]$; then

$$\mathcal{C} = \underbrace{(\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A) \cap \mathcal{C})}_{\mathcal{C}_1} \cup \underbrace{(v\mathcal{G}[B] \cap \mathcal{C})}_{\mathcal{C}_2}.$$

Let $\mathcal{B}_v$ be the B-component of v in $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$. If $\mathcal{C} \cap \mathcal{B}_v = \varnothing$ then $v\mathcal{G}[B] \cap \mathcal{C} = \varnothing$ and $\mathcal{C}$ coincides with some C-component of $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ which is a C-coset or a C-cluster. Suppose that $\mathcal{C} \cap \mathcal{B}_v \neq \varnothing$ and let w be a vertex of $\mathcal{C} \cap \mathcal{B}_v$. Then, on the one hand,

$$\mathcal{C} \cap v\mathcal{G}[B] = \mathcal{C} \cap w\mathcal{G}[B] = w\mathcal{G}[B \cap C]$$

while, on the other hand

$$\mathcal{C} \cap \mathcal{B}_v = w\mathcal{G}[B \cap C] \cap \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A).$$

The latter graph is the $(B \cap C)$-component of w in $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$, which happens to be a $(B \cap C)$-cluster or a $(B \cap C)$-coset contained in the C-component $\mathcal{C}$. Consequently, $\mathcal{C}$ is the union of the C-cluster $\mathcal{C} \cap \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ and the $(B \cap C)$-coset $w\mathcal{G}[B \cap C]$ whose intersection with $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ is the $(B \cap C)$-cluster $\mathcal{C} \cap \mathcal{B}_v$. Altogether, this just means that

$$\mathcal{C} = (\mathcal{C} \cap \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A))ⓦ\mathcal{G}[B \cap C],$$

that is, a $(B \cap C)$-augmented C-cluster, as required. $\square$

4. The two main technical results

In this section we prove the two main technical results. They will be essential to set up the inductive procedure to gain the series (3.3). In order to do so, we need another crucial definition. Assume, as above, that $|A| \geq 2$, that $G[A]$ is retractable and that $\mathcal{K} \subseteq \mathcal{G}[A]$ is admissible for ${}^{\subsetneq}A$-coset extension.

Definition 4.1. The full coset extension $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$ is *bridge-free* if

(1) the morphism $\iota_{\mathbb{P}_A} \colon \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A) \to \mathcal{G}[A]$ (Proposition 3.9) is an embedding;
(2) for every $B \subsetneq A$, if two vertices $u, v \in \mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A) \subseteq \mathcal{G}[A]$ (as per (1)) are B-connected in $\mathcal{G}[A]$ then they are B-connected even in $\mathsf{CE}(G, \mathcal{K}; \mathbb{P}_A)$.

In Theorem 4.2 below one has to take into account that, by Lemma 3.10, if for some group H, some subgraph $\mathcal{L} \subseteq \mathcal{H}[A]$ is admissible for ${}^{\subsetneq}A$-coset extension, then, for all $B \subsetneq A$, every B-component $v\mathcal{L}[B]$ is admissible for ${}^{\subsetneq}B$-coset extension and the morphism of Proposition 3.9 is an embedding $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B) \hookrightarrow v\mathcal{H}[B]$.

Theorem 4.2 (forward induction)**.** *Let H be an E-group, $A \subseteq E$, $|A| \geq 3$ and suppose that $H[A]$ is retractable. Let $\mathcal{L} \subseteq \mathcal{H}[A]$ be a connected A-graph which is admissible for ${}^{\subsetneq}A$-coset extension. Assume that for all $B \subsetneq A$ and every vertex $v \in \mathcal{L}$ the following hold:*

1. *the full ${}^{\subsetneq}B$-coset extension $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B)$ is bridge-free in $\mathcal{H}[B]$;*
2. *the full ${}^{\subsetneq}B$-coset extension $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B)$ has the cluster property.*

Then the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ has the cluster property.

Proof. Let $B \subsetneq A$, let $\mathcal{B}$ be a B-component of $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ and suppose that $\mathcal{B}$ has empty intersection with the skeleton $\mathcal{L}$. Consider first the case that no vertex of $\mathcal{B}$ has support of size smaller than $|A|-1$, and let (A_1, v_1) be some support of $\mathcal{B}$. That is, $A_1 \subsetneq A$, $|A_1| = |A| - 1$ and $v_1\mathcal{H}[A_1]$ has non-empty intersection with $\mathcal{B}$. We claim that, in this case, the entire component $\mathcal{B}$ is contained in the coset $v_1\mathcal{H}[A_1]$, hence $\mathcal{B}$ is a $(B \cap A_1)$-coset inside $v_1\mathcal{H}[A_1]$ and (A_1, v_1) is the unique minimal support of $\mathcal{B}$. Suppose this claim were not true. Then $\mathcal{B}$ has non-empty intersection with some constituent coset $v_2\mathcal{H}[A_2]$ for some $A_2 \neq A_1$ such that there exists a vertex $s_1 \in \mathcal{B} \cap v_1\mathcal{H}[A_1]$ which is connected with some vertex $s_2 \in (\mathcal{B} \cap v_2\mathcal{H}[A_2]) \setminus v_1\mathcal{H}[A_1]$ by some edge e. Since $s_2 \notin v_1\mathcal{H}[A_1]$ also $e \notin v_1\mathcal{H}[A_1]$. Then e belongs to a coset $v_3\mathcal{H}[A_3]$ (which perhaps coincides with $v_2\mathcal{H}[A_2]$) with $A_3 \neq A_1$. In any case, $s_1, s_2 \in v_3\mathcal{H}[A_3]$ (if a graph contains an edge then also its initial and terminal vertices). It follows that s_1 is supported by (A_3, v_3), that is, $s_1 \in v_1\mathcal{H}[A_1] \cap v_3\mathcal{H}[A_3] = v\mathcal{H}[A_1 \cap A_3]$ for some vertex v. Altogether this contradicts the assumption that no vertex of $\mathcal{B}$ has support of size smaller than $|A| - 1$.

We are left with the case that $\mathcal{B}$ admits support of size strictly smaller than $|A|-1$. It follows from the arguments of the preceding paragraph that every non-empty intersection of $\mathcal{B}$ with some constituent coset of $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ admits a vertex which is supported by less than $|A| - 1$ letters. Denote the constituent cosets of $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ having non-empty intersection with $\mathcal{B}$ by $v_1\mathcal{H}[A_1], \ldots, v_n\mathcal{H}[A_n]$, where $|A_i| = |A| - 1$ for all i. That is,

$$\mathcal{B} = \mathcal{B} \cap \left(\bigcup_{i=1}^{n} v_i\mathcal{H}[A_i] \right) = \bigcup_{i=1}^{n} (\mathcal{B} \cap v_i\mathcal{H}[A_i]) = \bigcup_{i=1}^{n} \mathcal{B}_i$$

for $\mathcal{B}_i = \mathcal{B} \cap v_i\mathcal{H}[A_i]$. Every $\mathcal{B}_i$ is a B_i-coset subgraph of $v_i\mathcal{H}[A_i]$ where $B_i = B \cap A_i$ and all B_i have size at most $|A|-2$. (If for some i, $|B_i| = |A|-1$ then $B_i = A_i$ and $\mathcal{B}_i = v_i\mathcal{H}[A_i]$ would have non-empty intersection with the skeleton $\mathcal{L}$.)

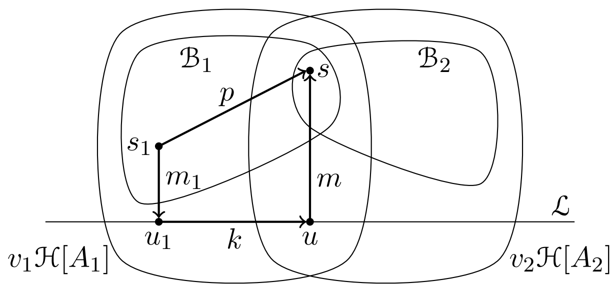

FIGURE 5

We need to verify items (1) and (2) of Definition 3.12. For $i = 1, \ldots, n$ denote by $\mathcal{A}_i$ the A_i-component $v_i\mathcal{L}[A_i]$ of v_i in $\mathcal{L}$. By Lemma 3.10, $\mathcal{A}_i$ is admissible for ${}^{\subsetneq}A_i$-coset extension and the full ${}^{\subsetneq}A_i$-coset extension $\mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i})$ embeds into $v_i\mathcal{H}[A_i]$ (via the mapping of Proposition 3.9). Since $\mathcal{B}_i$ admits vertices supported by fewer than $|A_i| = |A| - 1$ letters we have that $\mathcal{B}_i \cap \mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i}) \neq \varnothing$ — once more we take into account that

$$\mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i}) \subseteq v_i\mathcal{H}[A_i] \subseteq \mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A).$$

Bridge-freeness of $\mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i})$ (assumption (1)) implies that

$$\mathcal{B}_i \cap \mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i})$$

is connected. By assumption (2) therefore, $\mathcal{B}_i \cap \mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i})$ has unique minimal support in $\mathsf{CE}(H, \mathcal{A}_i; \mathbb{P}_{A_i})$, say (D_i, u_i). But then the pair (D_i, u_i) also provides unique minimal support of $\mathcal{B}_i$ in $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$. In particular, this means that in order to connect $\mathcal{B}_i$ with $\mathcal{L}$ we require (at least) a D_i-path which necessarily leads to the D_i-component $u_i\mathcal{A}_i[D_i]$. So, for every i, there exist vertices $s_i \in \mathcal{B}_i$, $u_i \in \mathcal{A}_i$ and a word $m_i \in \widetilde{D_i}^*$ labelling a path $s_i \longrightarrow u_i$ which runs entirely inside the coset $u_i\mathcal{H}[D_i]$ which in turn is contained in $v_i\mathcal{H}[A_i] = u_i\mathcal{H}[A_i]$.

Since $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i$ is connected there are i, j such that $\mathcal{B}_i \cap \mathcal{B}_j \neq \varnothing$; after some renumbering we may assume that $\mathcal{B}_1 \cap \mathcal{B}_2 \neq \varnothing$. Then also $v_1\mathcal{H}[A_1] \cap v_2\mathcal{H}[A_2] \neq \varnothing$ and thus $v_1\mathcal{H}[A_1] \cap v_2\mathcal{H}[A_2] = v\mathcal{H}[A_1 \cap A_2]$ for some $v \in \mathcal{A}_1 \cap \mathcal{A}_2$; notice that by Lemma 3.11, $\mathcal{A}_1 \cap \mathcal{A}_2 = v\mathcal{L}[A_1 \cap A_2]$ is an $(A_1 \cap A_2)$-component of $\mathcal{L}$. The intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ is a $B \cap A_1 \cap A_2$ coset in $v\mathcal{H}[A_1 \cap A_i]$. Similarly as for $\mathcal{B}_1$ one argues that $\mathcal{B}_1 \cap \mathcal{B}_2$ has unique minimal support in $\mathsf{CE}(H, \mathcal{A}_1 \cap \mathcal{A}_2; \mathbb{P}_{A_1 \cap A_2})$, (D, u) say, which (as for $\mathcal{B}_1$) provides unique minimal support of $\mathcal{B}_1 \cap \mathcal{B}_2$ in $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$. Let $s \in \mathcal{B}_1 \cap \mathcal{B}_2$ be a vertex which attains the support (D, u). So far, the situation is depicted in Figure 5. We note that $D \subseteq A_1 \cap A_2$ and so

$$u\mathcal{H}[D] \subseteq u\mathcal{H}[A_1 \cap A_2] = v_1\mathcal{H}[A_1] \cap v_2\mathcal{H}[A_1].$$

Since (D, u) is some support of $\mathcal{B}_1$ we have $D_1 \subseteq D$ and $u_1\mathcal{H}[D_1] \subseteq u\mathcal{H}[D]$. Hence there is a D-path $u_1 \longrightarrow u$ labelled k, say, which runs inside $\mathcal{A}_1$, and a D-path $u \longrightarrow s$ labelled m. Altogether, there is a D-path $s_1 \longrightarrow s$ labelled

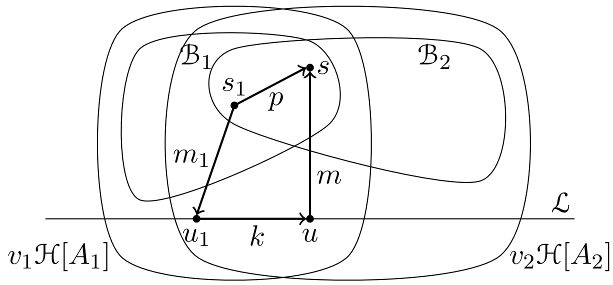

Figure 6

m_1km (this path runs entirely in $v_1\mathcal{H}[A_1]$). Since $s_1, s \in \mathcal{B}_1$ there is also a B_1-path $s_1 \longrightarrow s$ where $B_1 = B \cap A_1$, labelled p, say. Again, this path runs inside $v_1\mathcal{H}[A_1]$. Since $H[A_1]$ is retractable we have $[p]_{H[A_1]} = [p']_{H[A_1]}$ where p' is the word obtained from p by deletion of all letters not in D. Hence, there is a D-path $s_1 \longrightarrow s$ which runs entirely in $\mathcal{B}_1 \cap u\mathcal{H}[D]$ and, in particular, $s_1 \in u\mathcal{H}[D] \subseteq u\mathcal{H}[A_1 \cap A_2] = v_1\mathcal{H}[A_1] \cap v_2\mathcal{H}[A_2]$ so that $s_1 \in \mathcal{B}_1 \cap \mathcal{B}_2$. Since (D_1, u_1) supports s_1 and therefore also $\mathcal{B}_1 \cap \mathcal{B}_2$ it follows $D \subseteq D_1$ and therefore $D = D_1$ as the converse inclusion has been already shown. In particular, (D, u) provides unique minimal support of $\mathcal{B}_1$ which is attained at $s_1 \in \mathcal{B}_1 \cap \mathcal{B}_2$. So the configuration in Figure 5 really looks as depicted in Figure 6. By the same reasoning we obtain that $s_2 \in \mathcal{B}_1 \cap \mathcal{B}_2$ and $D_2 = D$. Altogether, $s_1, s_2 \in \mathcal{B}_1 \cap \mathcal{B}_2$ and (D, u) provides unique minimal support of $\mathcal{B}_1$ as well as $\mathcal{B}_2$, attained at s_1 was well as s_2. Now we continue by induction. Let $2 \leq k < n$ and suppose, subject to some renumbering of the cosets $\mathcal{B}_i$ we have already shown that $s_1, \ldots, s_k \in \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_k$ and all these $\mathcal{B}_i$ have unique minimal support (D, u) attained at all these s_i. Again there are $j \in \{1, \ldots, k\}$ and $i \in \{k+1, \ldots, n\}$ such that $\mathcal{B}_j \cap \mathcal{B}_i \neq \varnothing$ and after some renumbering we may assume that $j = k$ and $i = k+1$. Then, as for the case $k = 1$, $s_k, s_{k+1} \in \mathcal{B}_k \cap \mathcal{B}_{k+1}$ and the unique minimal support of $\mathcal{B}_k \cap \mathcal{B}_{k+1}$ is (D, u). Again, $s_k \in \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_k \cap \mathcal{B}_{k+1}$ and so $\mathcal{B}_j \cap \mathcal{B}_{k+1} \neq \varnothing$ for all $j \leq k$, therefore $s_j, s_{k+1} \in \mathcal{B}_j \cap \mathcal{B}_{k+1}$ and hence $s_1, \ldots, s_{k+1} \in \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_{k+1}$ and (D, u) provides unique minimal support for $\mathcal{B}_{k+1}$ attained at s_{k+1}. So $s_1, \ldots, s_n \in \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_n$ and $\bigcup_{i=1}^n \mathcal{B}_i$ has unique minimal support (D, u) attained at some elements of $\bigcap_{i=1}^n \mathcal{B}_i$; the union of any two cosets $\mathcal{B}_i \cup \mathcal{B}_j$ is contained in the A-cluster $v_i\mathcal{H}[A_i] \cup v_j\mathcal{H}[A_j]$, hence is itself a cluster. So the intersection $\mathcal{B}_i \cap \mathcal{B}_j$ of any two of these cosets is connected. Altogether, the union $\mathcal{B} = \bigcup \mathcal{B}_i$ is isomorphic with the cluster $\mathsf{CL}(H[B], \{B \cap A_i \mid i = 1, \ldots, n\})$. $\square$

The case $|A| = 2$ which is not handled in Theorem 4.2 is actually trivial.

Proposition 4.3. *Let H be an E-group, $A \subseteq E$ with $|A| = 2$ and $H[A]$ be retractable. Then every connected A-subgraph $\mathcal{L}$ is admissible for ${}^{\subsetneq}A$-coset extension and the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ has the cluster property.*

Proof. Definition 3.12 is fulfilled for trivial reasons: only the empty set $C = \varnothing$ satisfies $C \subsetneq B \subsetneq A$. Every constituent coset of $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ is of the form $v\mathcal{H}[a]$ for some letter $a \in A$. Hence, for $B \subsetneq A$, the only B-components of $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ which have empty intersection with $\mathcal{L}$ are singleton vertices which clearly have unique minimal support. $\square$

Combination of this with Theorem 4.2 implies the next result.

Corollary 4.4. *Let H be an E-group, $A \subseteq E$, $|A| \geq 3$ and suppose that $H[A]$ is retractable. Let $\mathcal{L} \subseteq \mathcal{H}[A]$ be a connected A-graph which is admissible for ${}^{\subsetneq}A$-coset extension. Assume that for all $B \subsetneq A$ and every vertex $v \in \mathcal{L}$ the full ${}^{\subsetneq}B$-coset extension $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B)$ embeds into $v\mathcal{H}[B]$ and is bridge-free; then the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ has the cluster property.*

Proof. This is by induction on $|A|$. For $|A| = 3$ we note that for any $B \subsetneq A$ we only need to consider $|B| = 2$, and by Proposition 4.3, $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B)$ has the cluster property. Hence the assumption of the corollary implies, by Theorem 4.2, that $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ has the cluster property. Now let $|A| > 3$ and suppose that claim is true for all B with $|B| < |A|$. In particular, $\mathsf{CE}(H, v\mathcal{L}[B]; \mathbb{P}_B)$ has the cluster property. Together with the assumption of the Corollary, Theorem 4.2 then implies that $\mathsf{CE}(H, \mathcal{L}; \mathbb{P}_A)$ has the cluster property. $\square$

Let $H \twoheadleftarrow G$ be an expansion of E groups and $\varphi \colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ be the induced canonical graph morphism. Let $\mathcal{L} \subseteq \mathcal{H}$ be a connected subgraph. A *cover of $\mathcal{L}$ in $\mathcal{G}$* (a $\mathcal{G}$-*cover* for short) is any connected component of the graph $\varphi^{-1}(\mathcal{L}) \subseteq \mathcal{G}$. Recall that a crucial feature of covers is the *path lifting property*: if $\mathcal{L}$ admits a path $u \longrightarrow v$ labelled $p \in \widetilde{E}^*$ and u' is any vertex of $\varphi^{-1}(\mathcal{L})$ such that $\varphi(u') = u$, then $\varphi^{-1}(\mathcal{L})$ admits a path labelled p with initial vertex u' (which path is mapped under φ onto the original path in $\mathcal{L}$).

Theorem 4.5 (upward induction)**.** *Let $1 \leq k < |E|$ and let H be an E-group which is $(k+1)$-retractable. Let $A \subseteq E$ with $|A| = k + 1$ and let $\mathcal{L}_H$ be a connected A-subgraph of $\mathcal{H}[A]$ such that*

(1) *$\mathcal{L}_H$ is admissible for ${}^{\subsetneq}A$-coset extension,*
(2) *the full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ has the cluster property.*

Let $H \twoheadleftarrow G$ be a k-stable expansion such that, for all $B \subsetneq A$ the trivial completion $\overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A) \textcircled{v} \mathcal{H}[B]}$ of every B-augmented full ${}^{\subsetneq}A$-coset extension $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ (thus, in particular $\overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$ itself) is the image of $\mathcal{G}$ under some canonical graph morphism. Then the following hold:

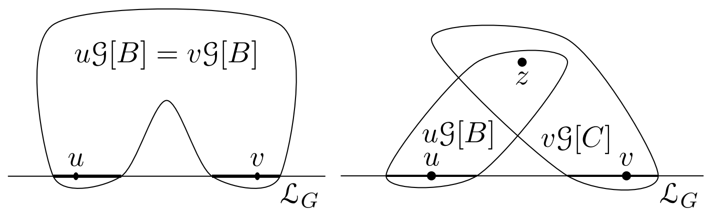

FIGURE 7

(i) *any cover $\mathcal{L}_G$ of $\mathcal{L}_H$ in $\mathcal{G}$ is admissible for $^{\subsetneq}\!A$-coset extension,*
(ii) *the full $^{\subsetneq}\!A$-coset extension $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ embeds into $\mathcal{G}[A]$,*
(iii) *the full $^{\subsetneq}\!A$-coset extension $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ is bridge-free.*

Proof. As for (i), that $\mathcal{L}_G$ is admissible for $^{\subsetneq}\!A$-coset extension follows from the fact that $\mathcal{L}_H$ is admissible for $^{\subsetneq}\!A$-coset extension and that the canonical morphism $G \twoheadrightarrow H$ is k-stable. In this case, the canonical morphism $\varphi\colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ is injective on B-components for $B \subsetneq A$ so that condition (3.4) is satisfied for $\mathcal{L}_G$ if it is satisfied for $\mathcal{L}_H = \varphi(\mathcal{L}_G)$.

Towards injectivity as required for (ii), let $\psi\colon \mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A) \to \mathcal{G}[A]$ be the canonical graph morphism of Proposition 3.9. We first show that for every $B \subsetneq A$ its restriction to $\mathsf{CE}(G, \mathcal{L}_G; B)$ is injective. Suppose this were not the case. Since the restriction to $\mathcal{L}_G$ is an embedding, that could only happen if two elements of two distinct constituent cosets $u\mathcal{G}[B]$ and $v\mathcal{G}[B]$ of $\mathsf{CE}(G, \mathcal{L}_G; B)$ were mapped onto each other and therefore both cosets $u\mathcal{G}[B]$ and $v\mathcal{G}[B]$ were mapped onto each other. The result in $\mathcal{G}[A]$ is depicted in Figure 7 (left-hand side). Take a canonical graph morphism $\varphi\colon \mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$. Since the expansion $G \twoheadrightarrow H$ is k-stable and $|B| \leq k$ the morphism φ maps $u\mathcal{G}[B] = v\mathcal{G}[B]$ isomorphically onto $\varphi(u)\mathcal{H}[B]$ and likewise onto $\varphi(v)\mathcal{H}[B]$, hence $\varphi(u)\mathcal{H}[B] = \varphi(v)\mathcal{H}[B]$ in $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ so that $\varphi(u)$ and $\varphi(v)$ are in the same B-component of $\mathcal{L}_H$. It follows that $\varphi(u)$ and $\varphi(v)$ can be connected by a B-path which runs in $\mathcal{L}_H$. However, that path could then be lifted to a B-path between u and v which runs in $\mathcal{L}_G$ since φ is bijective beween $u\mathcal{G}[B]$ and $\varphi(u)\mathcal{H}[B]$. This contradicts our assumption that the restriction $\psi \restriction \mathsf{CE}(G, \mathcal{L}_G; B)$ is not injective (Figure 7).

So it is sufficient to consider the case when vertices of distinct coset extension $\mathsf{CE}(G, \mathcal{L}_G; B)$ and $\mathsf{CE}(G, \mathcal{L}_G; C)$ are mapped onto each other. Let $B, C \subsetneq A$, $B \neq C$ and $x \in \mathsf{CE}(G, \mathcal{L}_G, B)$ and $y \in \mathsf{CE}(G, \mathcal{L}_G; C)$ be vertices such that $\psi(x) = \psi(y)$. We need to show that $x = y$ in $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ (that is, x and y both are in $\mathsf{CE}(G, \mathcal{L}_G; B \cap C)$ and coincide). From $\psi(x) = \psi(y)$ we see that in $\mathcal{G}[A]$ the situation is as depicted in Figure 7 (right hand side) with $\psi(x) = z = \psi(y)$. That is, u and z are connected by a B-path while v and z are connected by a C-path, and altogether $z \in u\mathcal{G}[B] \cap v\mathcal{G}[C]$. Let us consider some canonical graph morphism $\mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$ (according to the statement of the Theorem). It maps $\mathcal{L}_G$ onto $\mathcal{L}_H$, and let u', v', z' be the image vertices of u, v, z, respectively, under this morphism. Then

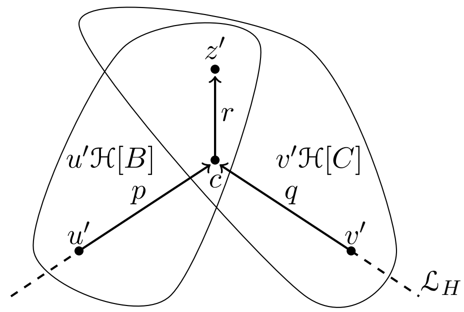

FIGURE 8

$u', v' \in \mathcal{L}_H$ and $z' \in u'\mathcal{H}[B] \cap v'\mathcal{H}[C]$. The latter intersection is a $(B \cap C)$-(constituent) coset of $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$, having non-empty intersection with the skeleton $\mathcal{L}_H$, say $u'\mathcal{H}[B] \cap v'\mathcal{H}[C] = c\mathcal{H}[B \cap C]$ for some $c \in \mathcal{L}_H$. Moreover, both intersections $\mathcal{L}_H \cap u'\mathcal{H}[B]$ and $\mathcal{L}_H \cap v'\mathcal{H}[C]$ are connected (namely B- respectively C-components of $\mathcal{L}_H$). The situation is depicted in Figure 8. So there are paths $u' \xrightarrow{p} c$ in $\mathcal{L}_H \cap u'\mathcal{H}[B]$, $v' \xrightarrow{q} c$ in $\mathcal{L}_H \cap v'\mathcal{H}[C]$ and $c \xrightarrow{r} z'$ in $u'\mathcal{H}[B] \cap v'\mathcal{H}[C]$. In particular, pr labels a path $u' \longrightarrow z'$, qr labels a path $v' \longrightarrow z'$. From k-stability of the expansion $G \twoheadrightarrow H$ it follows that the morphism $\mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$ is injective on all cosets $x\mathcal{G}[D]$ for all $D \subsetneq A$. In particular, this morphism is bijective between $u\mathcal{G}[B]$ and $u'\mathcal{H}[B]$ as well as between $v\mathcal{G}[C]$ and $v'\mathcal{H}[C]$. From this it follows that the paths in $u'\mathcal{H}[B] \cup v'\mathcal{H}[C]$ just mentioned lift to paths in $u\mathcal{G}[B] \cup v\mathcal{G}[C]$: hence there is a path $u \longrightarrow z$ labelled pr and one $v \longrightarrow z$ labelled qr. It follows that, in $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$,

$$u \cdot p = z \cdot r^{-1} = v \cdot q.$$

Since $p\colon u' \longrightarrow c$ runs in $\mathcal{L}_H$ and so does $q\colon v' \longrightarrow c$, the path $p\colon u \longrightarrow z \cdot r^{-1}$ runs in $\mathcal{L}_G$, and so does the path $q\colon v \longrightarrow z \cdot r^{-1}$. It follows that

$$u\mathcal{G}[B] = (z \cdot r^{-1})\mathcal{G}[B] \text{ and } v\mathcal{G}[C] = (z \cdot r^{-1})\mathcal{G}[C],$$

thus $u\mathcal{G}[B] \cap v\mathcal{G}[C] = (z \cdot r^{-1})\mathcal{G}[B \cap C]$ so that, in $\mathsf{CE}(G, \mathcal{L}_G; \{B, C\})$:

$$x = (z \cdot r^{-1}) \cdot r = y,$$

that is, x and y represent the same vertex in $\mathsf{CE}(G, \mathcal{L}_G; B \cap C)$, as required. Altogether, $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ embeds in $\mathcal{G}[A]$ via the morphism of Proposition 3.9.

It remains to argue for (iii): we need to show that $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ is bridge-free. So we have two vertices $v_1, v_2 \in \mathcal{L}_G$, $A_1, A_2 \subsetneq A$ and vertices $s_1 \in v_1\mathcal{G}[A_1]$, $s_2 \in v_2\mathcal{G}[A_1]$; in addition, for some $B \subsetneq A$ there is a B-path $s_1 \xrightarrow{p} s_2$ running in $\mathcal{G}[A]$ (all the following takes place in $\mathcal{G}[A]$) as depicted in Figure 9. In addition, there are an A-path $v_1 \xrightarrow{q} v_2$ running in $\mathcal{L}_G$ and A_i-paths (for $i = 1, 2$) $v_i \xrightarrow{f_i} s_i$ running in $v_i\mathcal{G}[A_i]$. We first consider the

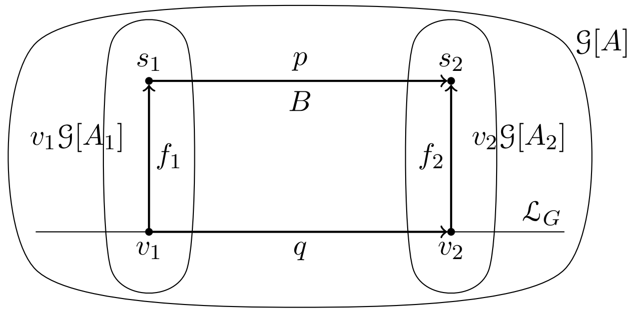

FIGURE 9

canonical graph morphism $\mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$ which maps $\mathcal{L}_G$ onto $\mathcal{L}_H$. Let $i \in \{1, 2\}$ and v_i' be the image of v_i in $\mathcal{L}_H$ under this morphism. The path $v_1 \xrightarrow{q} v_2$ is thereby mapped to the path $v_1' \xrightarrow{q} v_2'$. Next denote by s_i' the image of s_i; the path $v_i \xrightarrow{f_i} s_i$ running in $v_i\mathcal{G}[A_i]$ is mapped to the path $v_i' \xrightarrow{f_i} s_i'$ which runs in $v_i'\mathcal{H}[A_i]$. So far, these paths run in $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$. Further, the path $s_1 \xrightarrow{p} s_2$ is mapped to the path $s_1' \xrightarrow{p} s_2'$ which runs in $\overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$. It follows that there is a B-path $s_1' \xrightarrow{\dot{p}} s_2'$ which runs in $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ (in fact, $\dot{p}$ is the word obtained from p by deletion of the letters which traverse loop edges of $\overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)} \setminus \mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$).

So consider the B-component $\mathcal{B}$ of $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ which contains the two vertices s_1' and s_2'. The cluster property of $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ shows the following: either $\mathcal{B}$ has non-empty intersection with the skeleton $\mathcal{L}_H$, or else $\mathcal{B}$ is a B-cluster (the existence of unique minimal support is not needed in this context). Assume the latter case first: as a B-cluster $\mathcal{B}$ is the union $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ of $(B \cap C_i)$-cosets where $C_i \subsetneq A$, $|C_i| = |A| - 1$ and we may assume that $s_i' \in \mathcal{B}_i$ for $i = 1, 2$. The pairs (A_1, v_1') and (A_2, v_2') provide support for s_1' and s_2', respectively, and $\mathcal{B}_1 \subseteq v_1'\mathcal{H}[C_1]$ and $\mathcal{B}_2 \subseteq v_2'\mathcal{H}[C_2]$, and the two cosets $\mathcal{B}_1 = s_1'\mathcal{H}[C_1 \cap B]$ and $\mathcal{B}_2 = s_2'\mathcal{H}[C_2 \cap B]$ have non-empty intersection (indeed, $\mathcal{B}_1 \cap \mathcal{B}_2$ contains the core of $\mathcal{B}$). Hence $v_1'\mathcal{H}[C_1] \cap v_2'\mathcal{H}[C_2] \neq \varnothing$ so that $v_1'\mathcal{H}[C_1] \cap v_2'\mathcal{H}[C_2] = v\mathcal{H}[C]$ for $C = C_1 \cap C_2$ and some vertex $v \in \mathcal{L}_H$. The situation is depicted in Figure 10. In particular, there is a vertex $s \in s_1'\mathcal{H}[B \cap C_1] \cap s_2'\mathcal{H}[B \cap C_2]$ and there are $B \cap C_i$-paths

$$s_1' \xrightarrow{p_1} s \xrightarrow{p_2} s_2'$$

labelled p_i $(i = 1, 2)$. We now consider the B-augmentation of $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ at the vertex s and the canonical graph morphism

$$\psi \colon \mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A) \circledS \mathcal{H}[B]}$$

mapping $\mathcal{L}_G$ onto $\mathcal{L}_H$. The graphs $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ and $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A) \circledS \mathcal{H}[B]$ very much look the same except that the cluster $\mathcal{B}$ in $\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)$ is blown up to the full coset $s\mathcal{H}[B]$ in the latter graph. The morphism ψ now maps the path $s_1 \xrightarrow{p} s_2$ to the path $s_1' \xrightarrow{p} s_2'$ which runs in $s\mathcal{H}[B]$; but

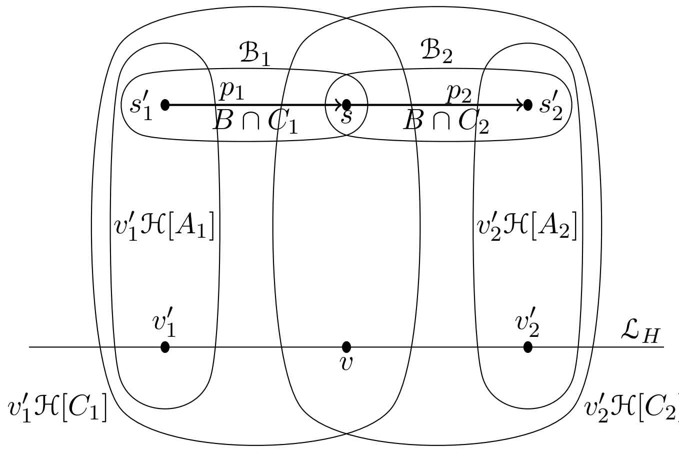

FIGURE 10

$s_1' \xrightarrow{p_1} s \xrightarrow{p_2} s_2'$ also run in $s\mathcal{H}[B]$ which implies that $[p]_H = [p_1p_2]_H$. Since the expansion $G \twoheadrightarrow H$ is k-stable and $|B| \leq k$, it follows that $[p]_G = [p_1p_2]_G$. But then, the path $s_1 \xrightarrow{p_1p_2} s_2$ runs entirely in $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$ and thus provides a B-path between s_1 and s_2 in the coset extension $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$. This follows from the fact that the path $s_1 \xrightarrow{p_1} s \cdot p_1$ runs in $v_1\mathcal{G}[C_1]$ while $s_1 \cdot p_1 \xrightarrow{p_2} s_1 \cdot p_1p_2 = s_2$ runs in $v_2\mathcal{G}[C_2]$ since ψ provides isomorphisms $v_1\mathcal{G}[C_1] \twoheadrightarrow v_1'\mathcal{H}[C_1]$ and $v_2\mathcal{G}[C_2] \twoheadrightarrow v_2'\mathcal{H}[C_2]$.

Now consider the (first) case when $\mathcal{B}$ has non-empty intersection with the skeleton $\mathcal{L}_H$. In this case $\mathcal{B}$ is a full B-coset $\mathcal{B} = v\mathcal{H}[B]$ for some vertex $v \in \mathcal{L}_H$. The canonical morphism $\varphi\colon \mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H, \mathcal{L}_H; \mathbb{P}_A)}$ induces an isomorphism between $v\mathcal{H}[B]$ and $\varphi^{-1}(v\mathcal{H}[B]) = \varphi^{-1}(v)\mathcal{G}[B] = s_1\mathcal{G}[B] = s_2\mathcal{G}[B]$. But $\varphi^{-1}(v) \in \mathcal{L}_G$ so that $s_1\mathcal{G}[B] = s_2\mathcal{G}[B]$ is contained in $\mathsf{CE}(G, \mathcal{L}_G; \mathbb{P}_A)$. □

5. Construction of the group G

The group G announced in Lemma 2.8 will be constructed via a series of expansions

$$G_1 \twoheadleftarrow H_1 \twoheadleftarrow G_2 \twoheadleftarrow \cdots \twoheadleftarrow G_{|E|-1} \twoheadleftarrow H_{|E|-1} \twoheadleftarrow G_{|E|} = G \tag{5.1}$$

where, for every k, the expansions $G_k \twoheadleftarrow H_k$ and $H_k \twoheadleftarrow G_{k+1}$ are k-stable and the groups H_k and G_{k+1} are $(k+1)$-retractable. Compare the series (3.3) above, now interleaved with the intermediate stages H_k in (5.1). Each group in this series is the transition group of a certain graph,

$$G_k = \mathscr{T}(\mathcal{X}_k) \text{ and } H_k = \mathscr{T}(\mathcal{Y}_k),$$

where $\mathcal{Y}_k$ is obtained from $\mathcal{X}_k$ by adding certain connected components, and similarly $\mathcal{X}_{k+1}$ from $\mathcal{Y}_k$. The idea of this iterative procedure is as follows. The given oriented graph $\mathcal{E} = (V \cup \widetilde{E}; \alpha, \omega, {}^{-1})$ is considered as an E-labelled graph where every edge gets its own label and $\mathcal{X}_1$ is a certain completion of it. Suppose that for $k \geq 1$ the graph $\mathcal{X}_k$ and therefore its transition group G_k have already been constructed. The step $\mathcal{X}_k \rightsquigarrow \mathcal{Y}_k$, and hence the step $G_k \rightsquigarrow H_k$, raises the "degree of retractability" from k to $k+1$ and thereby lays the ground for the transition $H_k \rightsquigarrow G_{k+1}$. This step is intended to ensure the following: suppose that p is a word over $k+1$ letters which forms a path $u \longrightarrow v$ in $\mathcal{E}$ and $a \in \mathrm{co}(p)$ for some $a \in E$; if H_k satisfies the relation $p = p_{a\to 1}$ and there is no word q in the letters $\mathrm{co}(p) \setminus \{a\}$ labelling a path $u \longrightarrow v$ in $\mathcal{E}$ and such that H_k satisfies the relation $p_{a\to 1} = q$ then some component of $\mathcal{X}_{k+1} \setminus \mathcal{Y}_k$ guarantees that G_{k+1} avoids the relation $p = p_{a\to 1}$.

5.1. **Definition of G_1 and the transition $G_k \rightsquigarrow H_k$.** The idea of the construction of the graph $\mathcal{X}_1$ is to extend the given oriented graph $\mathcal{E} = (V \cup \widetilde{E}, \alpha, \omega, ^{-1})$ to a complete E-graph on the vertex set V in whose transition group the permutation $[e]$ corresponding to any non-loop edge e is the transposition of V that swaps the two vertices αe and ωe. Let $\mathcal{E} = (V \cup \widetilde{E}; \alpha, \omega, ^{-1})$ be a finite connected oriented graph. We let the set of positive edges E be our alphabet and label every edge e by itself. Thereby we get the E-labelled graph $(V \cup \widetilde{E}; \alpha, \omega, ^{-1}, \ell, E)$ where ℓ is the identity function mapping every $e \in \widetilde{E}$, considered as an edge, to itself, considered as a label. The resulting graph is even an E-graph for trivial reasons, since every label appears exactly once.

Next, for every non-loop edge e we add a new edge $\bar{e}$ and set

$$\alpha\bar{e} := \omega e,\ \omega\bar{e} := \alpha e,\ \ell(\bar{e}) := \ell(e) = e.$$

We have thus completed every non-loop edge $u \xrightarrow[e]{} v$ to a 2-cycle $u \underset{e}{\overset{e}{\rightleftarrows}} v$.

Let us denote the set of all positve edges so obtained (the original ones and the added ones) by F; then the oriented E-graph $\mathcal{F} = (V \cup \widetilde{F}; \alpha, \omega, ^{-1}, \ell, E)$ is weakly complete. Let $\mathcal{X}_1 := \overline{\mathcal{F}}$ be its trivial completion. The transition group $G_1 := \mathscr{T}(\mathcal{X}_1)$ is an E-generated group of permutations acting on the vertex set V. For every $e \in E$, $[e]_{G_1}$ is either a transposition (if e is not a loop edge then $[e]$ swaps αe and ωe) or the identity permutation (if e is a loop edge). Note that two distinct labels $e, f \in E$ may represent the same permutation of V (since we allow multiple edges in $\mathcal{E}$).

Remark 5.1. Instead of completing all non-loop edges to 2-cycles we could equally well complete every such edge e to an n-cycle for any fixed $n \geq 2$, by attaching to the edge $u \xrightarrow{e} v$ an e-path $u \xleftarrow{e} \cdots \xleftarrow{e} v$ consisting of a sequence of $n-1$ new edges labelled e and $n-2$ new intermediate vertices. In the resulting transition group, the permutation $[e]$ assigned to e then is a cyclic permutation of length n mapping αe to ωe. Distinct labels coming from non-loop edges then automatically represent different permutations provided that $n \geq 3$.

The transition from G_k to H_k is easily described. Suppose we have already defined the graph $\mathcal{X}_k$ and thus the group $G_k = \mathscr{T}(\mathcal{X}_k)$. We set

$$\mathcal{Y}_k := \mathcal{X}_k \sqcup \bigsqcup_{\substack{A \subseteq E \\ |A|=k}} \overline{\mathcal{G}_k[A]}.$$

Provided that G_k is k-retractable, this expansion is k-stable and $H_k = \mathscr{T}(\mathcal{Y}_k)$ is $(k{+}1)$-retractable (Theorem 3.2). In particular, H_1 is 2-retractable.

5.2. **The transition** $H_k \rightsquigarrow G_{k+1}$**.** The expansion $H_k \twoheadleftarrow G_{k+1}$ is more delicate. We let $\mathcal{X}_{k+1} = \mathcal{Y}_k \sqcup \overline{\mathcal{Z}_k}$ where $\mathcal{Z}_k$ is a weakly complete graph whose components we shall define now. As inductive hypothesis on G_k we assume that

(i) G_k is k-retractable

and, for every $B \subseteq E$ with $|B| \leq k$, for the $\mathcal{G}_k$-cover $\mathcal{C}_{\mathcal{G}_k}$ of every connected component $\mathcal{C}$ of $\mathcal{B} = \langle B \rangle$ the following hold:

(ii) $\mathcal{C}_{\mathcal{G}_k}$ is admissible for $^{\subsetneq}B$-coset extension,
(iii) the full $^{\subsetneq}B$-coset extension $\mathsf{CE}(G_k, \mathcal{C}_{\mathcal{G}_k}; \mathbb{P}_B)$ embeds into $\mathcal{G}_k[B]$,
(iv) the full $^{\subsetneq}B$-coset extension $\mathsf{CE}(G_k, \mathcal{C}_{\mathcal{G}_k}; \mathbb{P}_B)$ is bridge-free.

By (i) and Theorem 3.2, $H_k \twoheadrightarrow G_k$ is k-stable. Let $\psi_k \colon \mathcal{G}_k \twoheadrightarrow \mathcal{X}_1$ be some canonical graph morphism, $\chi_k \colon \mathcal{H}_k \twoheadrightarrow \mathcal{G}_k$ the graph morphism induced by the canonical morphism $H_k \twoheadrightarrow G_k$, and let $\varphi_k = \psi_k \circ \chi_k$. Note that χ_k is injective on connected B-subgraphs for $|B| \leq k$ (by k-stability).

Let $A \subseteq E$, which is a subset of (positive) edges of $\mathcal{E} \subseteq \mathcal{X}_1$; assume that $|A| = k+1$ and let $\mathcal{A} = \langle A \rangle$ be the subgraph of $\mathcal{E}$ spanned by A. Let $\mathcal{C}$ be a connected component of $\mathcal{A}$ and $\mathcal{C}_{\mathcal{H}_k}$ be an $\mathcal{H}_k$-cover of $\mathcal{C}$, that is, some connected component of $\varphi_k^{-1}(\mathcal{C})$. We intend to show that $\mathcal{C}_{\mathcal{H}_k}$ is admissible for $^{\subsetneq}A$-coset extension (with respect to H_k) and that $\mathsf{CE}(H_k, \mathcal{C}_{\mathcal{H}_k}; \mathbb{P}_A)$ has the cluster property.

Let $B \subsetneq A$ and let $\mathcal{U} \subseteq \mathcal{C}_{\mathcal{H}_k}$ be some B-component of $\mathcal{C}_{\mathcal{H}_k}$. Then $\mathcal{U}' := \varphi_k(\mathcal{U}) \subseteq \mathcal{C}$ is a B-component of $\mathcal{C}$ and hence is a connected component of $\mathcal{B} := \langle B \rangle$. By the inductive hypothesis, any $\mathcal{G}_k$-cover $\mathcal{U}'_{\mathcal{G}_k}$ of $\mathcal{U}'$ is admissible for $^{\subsetneq}B$-coset extension (with respect to G_k) and $\mathsf{CE}(G_k, \mathcal{U}'_{\mathcal{G}_k}; \mathbb{P}_B)$ embeds into $\mathcal{G}_k[B]$ and is bridge-free. Since the morphism $\chi_k \colon \mathcal{H}_k \twoheadrightarrow \mathcal{G}_k$ is injective on B-components (that is, injective on B-cosets) it follows that $\mathcal{U}'_{\mathcal{G}_k} \cong \mathcal{U}$ and hence also

$$\mathsf{CE}(G_k, \mathcal{U}'_{\mathcal{G}_k}; \mathbb{P}_B) \cong \mathsf{CE}(H_k, \mathcal{U}; \mathbb{P}_B). \tag{5.2}$$

As the latter graph embeds into $\mathcal{H}_k[B] \cong \mathcal{G}_k[B]$, it follows that condition (3.4) of Definition 3.7 is fulfilled so that $\mathcal{C}_{\mathcal{H}_k}$ is admissible for $^{\subsetneq}A$-coset extension (with respect to H_k). Once more by the inductive hypothesis, every graph in (5.2) is bridge-free. Then, by Corollary 4.4, the full $^{\subsetneq}A$-coset extension $\mathsf{CE}(H_k, \mathcal{C}_{\mathcal{H}_k}; \mathbb{P}_A)$ itself has the cluster property. We can now define the components of the graph $\mathcal{Z}_k$.

Definition 5.2. The graph $\mathcal{Z}_k$ is the disjoint union of

(1) all augmented A-clusters

$$\mathsf{CL}(H_k[A],\mathbb{P})\textcircled{v}\mathcal{H}_k[B]$$

for $A \subseteq E$ with $|A| = k+1$, $\mathbb{P}$ a set of proper subsets of A, v a vertex of $\mathsf{CL}(H_k[A],\mathbb{P})$ and $B \subsetneq A$;

(2) all augmented full ${}^{\subsetneq}\!A$-coset extensions

$$\mathsf{CE}(H_k,\mathcal{C}_{\mathcal{H}_k};\mathbb{P}_A)\textcircled{v}\mathcal{H}_k[B]$$

for $A \subseteq E$ with $|A| = k+1$, $\mathcal{C}$ a connected component of $\mathcal{A} = \langle A\rangle$, $\mathcal{C}_{\mathcal{H}_k}$ an $\mathcal{H}_k$-cover of $\mathcal{C}$, $\mathbb{P}_A$ the set of all proper subsets of A, v a vertex of $\mathsf{CE}(H_k,\mathcal{C}_{\mathcal{H}_k};\mathbb{P}_A)$ and $B \subsetneq A$.

We note that the augmented clusters and augmented coset extensions contain, for $B = \varnothing$, all "plain" clusters and coset extensions. Recall that $G_{k+1} = \mathscr{T}(\mathcal{X}_{k+1}) = \mathscr{T}(\mathcal{Y}_k \sqcup \overline{\mathcal{Z}_k})$.

Proposition 5.3. *The expansion $H_k \twoheadleftarrow G_{k+1}$ is k-stable and hence G_{k+1} is $(k+1)$-retractable.*

Proof. We need to prove k-stability, the second assertion then follows from Theorem 3.2 by inductive hypothesis (i) and the definition of H_k. Let $C \subseteq E$ with $|C| = k$, let $p \in \widetilde{C}^*$ and assume that $[p]_{G_{k+1}} \neq 1$; we need to show that $[p]_{H_k} \neq 1$. There exists a component $\mathcal{L}$ of $\mathcal{Y}_k$ or of $\overline{\mathcal{Z}_k}$ witnessing the inequality $[p]_{G_{k+1}} \neq 1$. That is, in this component there is a vertex v such that $v \cdot p \neq v$. If the witnessing component $\mathcal{L}$ belongs to $\mathcal{Y}_k$, then we are done since then $[p]_{H_k} \neq 1$ immediately follows as $H_k = \mathscr{T}(\mathcal{Y}_k)$. If $\mathcal{L}$ is a component of $\overline{\mathcal{Z}_k}$, then $\mathcal{L} = \overline{\mathcal{M}}$ where $\mathcal{M}$ is of the form (1) or (2) of Definition 5.2 and the path $p\colon v \longrightarrow v \cdot p$ runs in the C-component $v\overline{\mathcal{M}}[C]$. Recall that $v\overline{\mathcal{M}}[C]$ denotes the C-component of v in the graph $\overline{\mathcal{M}}$ while $\overline{v\mathcal{M}[C]}$ is the trivial completion of $v\mathcal{M}[C]$, that is, the trivial completion of the C-component of v in $\mathcal{M}$. We have the inclusions

$$v\mathcal{M}[C] \subseteq v\overline{\mathcal{M}}[C] \subseteq \overline{v\mathcal{M}[C]},$$

where the latter two graphs differ only in loop edges having labels not in C. Hence C-paths in $v\overline{\mathcal{M}}[C]$ and $\overline{v\mathcal{M}[C]}$ traverse the same edges and meet the same vertices. It is therefore sufficient to look at $\overline{v\mathcal{M}[C]}$ instead of $v\overline{\mathcal{M}}[C]$. From Corollaries 3.4, 3.6 and Proposition 3.14, and since the coset extensions of Definition 5.2 (2) have the cluster property, it follows that, for the graph $\mathcal{M}$ in question, the C-component $v\mathcal{M}[C]$ must be isomorphic to one of the following:

(i) a full C-coset $\mathcal{H}_k[C]$, or

(ii) a C-cluster $\mathsf{CL}(H_k[C],\mathbb{P})$ for some set $\mathbb{P}$ of proper subsets of C (this includes for $\mathbb{P} = \{B\}$ also B-cosets $\mathcal{H}_k[B]$ for $B \subsetneq C$), or

(iii) a D-augmented C-cluster $\mathsf{CL}(H_k[C],\mathbb{P})\textcircled{w}\mathcal{H}_k[D]$ for some set $\mathbb{P}$ of proper subsets of C and D some proper subset of C.

In case (i), $v\mathcal{M}[C] \cong \mathcal{H}_k[C]$, the claim $[p]_{H_k} \neq 1$ again follows immediately. In cases (ii) or (iii) all members of the set $\mathbb{P}$ involved are proper subsets of C, hence have at most $k-1$ elements; then

$$\begin{aligned} & v\mathcal{M}[C] \cong \mathsf{CL}(H_k[C],\mathbb{P}) \cong \mathsf{CL}(G_k[C],\mathbb{P}) \cong \mathsf{CL}(H_{k-1}[C],\mathbb{P}) \\ \text{or}\quad & v\mathcal{M}[C] \cong \mathsf{CL}(H_k[C],\mathbb{P}) @ \mathcal{H}_k[D] \cong \mathsf{CL}(G_k[C],\mathbb{P}) @ \mathcal{G}_k[D] \\ & \qquad \cong \mathsf{CL}(H_{k-1}[C],\mathbb{P}) @ \mathcal{H}_{k-1}[D] \end{aligned}$$

imply that $\overline{v\mathcal{M}[C]}$ is isomorphic with a component of $\overline{\mathcal{Z}_{k-1}}$ so that $[p]_{G_k} \neq 1$, from which again $[p]_{H_k} \neq 1$ follows. $\square$

From Theorem 4.5 it follows that for every set $A \subseteq E$ with $|A| = k+1$ and every connected component $\mathcal{C}$ of $\mathcal{A}$, every $\mathcal{G}_{k+1}$-cover $\mathcal{C}_{\mathcal{G}_{k+1}}$ (that is, every connected component of $\psi_{k+1}^{-1}(\mathcal{C})$ in $\mathcal{G}_{k+1}$ where $\psi_{k+1} \colon \mathcal{G}_{k+1} \twoheadrightarrow \mathcal{X}_1$ is a canonical graph morphism) is admissible for $\subsetneq A$-coset extension, and the full $\subsetneq A$-coset extension $\mathsf{CE}(G_{k+1}, \mathcal{C}_{\mathcal{G}_{k+1}}; \mathbb{P}_A)$ embeds into $\mathcal{G}_{k+1}[A]$ and is bridge-free. If $|A| = l < k+1$ we have by induction that, for every connected component $\mathcal{C}$ of $\mathcal{A}$ the full $\subsetneq A$-coset extension $\mathsf{CE}(G_l, \mathcal{C}_{\mathcal{G}_l}; \mathbb{P}_A)$ embeds into $\mathcal{G}_l[A]$. But the expansion $G_l \twoheadleftarrow G_{k+1}$ is l-stable whence $\mathsf{CE}(G_l, \mathcal{C}_{\mathcal{G}_l}; \mathbb{P}_A) \cong \mathsf{CE}(G_{k+1}, \mathcal{C}_{\mathcal{G}_{k+1}}; \mathbb{P}_A)$ and $\mathcal{G}_l[A] \cong \mathcal{G}_{k+1}[A]$. Altogether we see that

(i) G_{k+1} is $(k+1)$-retractable (by Proposition 5.3)

and, for every $A \subseteq E$ with $|A| \leq k+1$, for the $\mathcal{G}_{k+1}$-cover $\mathcal{C}_{\mathcal{G}_{k+1}}$ of every connected component $\mathcal{C}$ of $\mathcal{A} = \langle A \rangle$ the following hold:

(ii) $\mathcal{C}_{\mathcal{G}_{k+1}}$ is admissible for $\subsetneq A$-coset extension,
(iii) the full $\subsetneq A$-coset extension $\mathsf{CE}(G_{k+1}, \mathcal{C}_{\mathcal{G}_{k+1}}; \mathbb{P}_A)$ embeds into $\mathcal{G}_{k+1}[A]$,
(iv) the full $\subsetneq A$-coset extension $\mathsf{CE}(G_{k+1}, \mathcal{C}_{\mathcal{G}_{k+1}}; \mathbb{P}_A)$ is bridge-free.

We have thus verified that G_{k+1} satisfies the conditions inductively assumed for G_k. In addition, the base case for this inductive procedure also works. The group H_1 is 2-retractable and so is G_2 since $G_2 \twoheadrightarrow H_1$ is 1-stable. By Proposition 4.3, for every set $A \subseteq E$ with $|A| = 2$, every H_1-cover $\mathcal{C}_{\mathcal{H}_1}$ of every component $\mathcal{C}$ of $\mathcal{A}$ is admissible for $\subsetneq A$-coset extension (with respect to H_1) and $\mathsf{CE}(H_1, \mathcal{C}_{\mathcal{H}_1}; \mathbb{P}_A)$ has the cluster property. Theorem 4.5 then implies that the $\mathcal{G}_2$-cover $\mathcal{C}_{\mathcal{G}_2}$ is admissible for $\subsetneq A$-coset extension (with respect to G_2) and that $\mathsf{CE}(G_2, \mathcal{C}_{\mathcal{G}_2}; \mathbb{P}_A)$ embeds in $\mathcal{G}_2[A]$ and is bridge-free (the assertions for G_2 can also be checked by direct inspection). Altogether the series of expansions

$$G_1 \twoheadleftarrow H_1 \twoheadleftarrow G_2 \twoheadleftarrow \cdots \twoheadleftarrow G_{|E|-1} \twoheadleftarrow H_{|E|-1} \twoheadleftarrow G_{|E|}$$

is well defined and $G = G_{|E|}$ is retractable.

5.3. **Properties of $G = G_{|E|}$.** We need to argue that G satisfies the requirements of Lemma 2.8. Requirement (2), that G is retractable, and therefore has a content function by Proposition 3.1, has already been proved. We are left with showing requirement (3), that the G-content of every word which labels a path $u \longrightarrow v$ in $\mathcal{E}$ spans a connected graph containing the vertices u and v or $u = v$ in case of empty content, and requirement (1), that every permutation of E induced by an automorphism of $\mathcal{E}$ extends to an automorphism of G. In this context, by "automorphism of $\mathcal{E}$" we mean automorphism of the *unlabelled oriented* graph $\mathcal{E} = (V \cup \widetilde{E}; \alpha, \omega, {}^{-1})$. Recall from the definition of an automorphism of an oriented graph that every such automorphism of $\mathcal{E}$ is required to induce a permutation on the set E of positive edges of $\mathcal{E}$, hence induces a permutation on our labelling alphabet E. Similarly, "automorphism of G" means automorphism of the mere group G (rather than G as an E-group). We start with item (1); (3) will then be dealt with in Lemma 5.5 and Corollary 5.6.

Proposition 5.4. *Every permutation $E \to E$ induced by an automorphism of the oriented graph $\mathcal{E}$ extends to an automorphism of G.*

Proof. Let γ be a permutation of E induced by an automorphism of $\mathcal{E}$, also denoted γ. We demonstrate the required property for all G_k and H_k, by induction on k. First note that γ (uniquely) extends to an automorphism $\hat{\gamma}$ of $\mathcal{X}_1$ from which the claim follows for the group G_1. Indeed, for every pair of vertices $u, v \in \mathcal{X}_1$ and every word $p \in \widetilde{E}^*$, we have $p\colon u \longrightarrow v$ if and only ${}^{\gamma}p\colon {}^{\hat{\gamma}}u \longrightarrow {}^{\hat{\gamma}}v$. Consequently, for every word $p \in \widetilde{E}^*$, G_1 satisfies the relation $p = 1$ if and only if it satisfies ${}^{\gamma}p = 1$.

So, let $k \geq 1$ and assume inductively that γ extends to an automorphism $\hat{\gamma}$ of $\mathcal{X}_k$ (this means that there is an automorphism $\hat{\gamma}$ of the oriented graph $\mathcal{X}_k$ such that for every edge $e \in \mathcal{X}_k$ we have $\ell({}^{\hat{\gamma}}e) = {}^{\gamma}\ell(e)$); by the same reasoning as for $k = 1$ we see that in this case γ extends to an automorphism of G_k. From the definition of the graph $\mathcal{Y}_k$ it now follows that γ extends to an automorphism $\hat{\gamma}$ of $\mathcal{Y}_k$ which again implies that γ extends to an automorphism of H_k. From this in turn it follows that γ extends to an automorphism of $\mathcal{X}_{k+1}$ and therefore again to an automorphism of G_{k+1}. $\square$

The assertion of the last proposition is essentially a direct consequence of the fact that the entire process behind our construction of G, on the basis of the given oriented graph $\mathcal{E}$, is symmetry-preserving. Indeed, none of the intermediate steps involves any choices that could possibly break symmetries in the input data, i.e. could be incompatible with isomorphisms between oriented input graphs $\mathcal{E}$. In particular, the inductive construction steps reflected in Theorems 4.2 and 4.5, proceed by cardinality of subsets of E and treat all subsets of the same size uniformly and in parallel.[2] Any isomorphism between oriented graphs $\mathcal{E} \cong \mathcal{E}'$ would successively extend to isomorphisms between the associated graphs $\mathcal{X}_i \cong \mathcal{X}_i'$ and $\mathcal{Y}_i \cong \mathcal{Y}_i'$ and induced isomorphisms between their transition groups $G_i \cong G_i'$ and $H_i \cong H_i$. In this sense, the entire inductive process underlying the expansion chain (5.1) is isomorphism-respecting, hence in particular compatible with permutations of E stemming from automorphisms of $\mathcal{E}$.

[2]This should be contrasted e.g. with constructions based on some enumeration of E, which could well break symmetries.

Finally, we have to deal with requirement (3) of Lemma 2.8. Recall that for a word $p \in \widetilde{E}^*$, $\mathrm{co}(p)$ is the set of all letters $a \in E$ for which a or a^{-1} occurs in p. We know that G is retractable hence for $g \in G$ we define the *content* $\mathrm{C}(g)$ of g by setting

$$\mathrm{C}(g) := \bigcap_{\substack{p \in \widetilde{E}^* \\ [p]_G = g}} \mathrm{co}(p).$$

Then every $g \in G$ has a representation $g = [q]_G$ which uses exactly the letters from $\mathrm{C}(g)$ (and/or their inverses), that is, $\mathrm{C}(g) = \mathrm{co}(q)$. We shall now prove that for a word $p \in \widetilde{E}^*$ which forms a path $u \longrightarrow v$ in $\mathcal{E}$ the content $\mathrm{C}([p]_G)$ (if non-empty) spans a connected graph containing the vertices u and v. The following lemma is crucial for this.

Lemma 5.5. *Let $p \in \widetilde{E}^*$ be a word that forms a path $u \longrightarrow v$ in $\mathcal{E}$; let $A = \mathrm{co}(p)$ and suppose that for some letter $a \in A$ and $B = A \setminus \{a\}$ there exists a word $r \in \widetilde{B}^*$ such that $[p]_G = [r]_G$. Then there exists a word $q \in \widetilde{B}^*$ such that $[p]_G = [q]_G$ and, in addition, q forms a path $u \longrightarrow v$ in $\mathcal{E}$.*

Proof. If p contains only loop edges then $u = v$ and the path meets only the vertex u. Moreover, $[p]_G = 1$ so that for q we may choose the empty word 1 which labels the empty path $u \longrightarrow u$ and $[q]_G = [1]_G$. Let us consider the case that p contains non-loop edges. If e is not a loop edge then no power e^n and e^{-n} for $n \geq 2$ forms a path; therefore, if $|A| = 1$ the only possibilities for p are $f(f^{-1}f)^n$ and $(ff^{-1})^{n+1}$ for $n \geq 0$ and $f \in \{e, e^{-1}\}$. In these cases the claim is obvious.

So, let $|A| = k + 1$, and let $\mathcal{A} = \langle A \rangle = \langle \pi_u^{\mathcal{E}}(p) \rangle$ be the subgraph of $\mathcal{E}$ spanned by A (which is connected) and let $\varphi_k \colon \mathcal{H}_k \twoheadrightarrow \mathcal{X}_1$ be the canonical morphism mapping $1 \in \mathcal{H}_k$ to u; let $\mathcal{A}_k \subseteq \mathcal{H}_k$ be the cover of $\mathcal{A}$ in $\mathcal{H}_k$ with $1 \in \mathcal{A}_k$ (that is, the connected component of $\varphi_k^{-1}(\mathcal{A})$ which contains the vertex 1). The path $\pi_u^{\mathcal{E}}(p)$ in $\mathcal{E}$ lifts to the path $\pi_1^{\mathcal{A}_k}(p)$. In particular, in $\mathcal{A}_k$ there is a p-labelled path starting at 1. We consider the full $\subsetneq A$-coset extension $\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)$ and note that $\mathsf{CE}(H_k, \mathcal{A}_k; B)$ is a subgraph of it. We also have the path $\pi_1^{\mathcal{G}}(p)$ in $\mathcal{G}$ starting at 1 and being labelled p. The canonical morphism $\psi_k \colon \mathcal{G} \twoheadrightarrow \overline{\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)}$ (mapping $1 \in \mathcal{G}$ to $1 \in \mathcal{A}_k$) maps $\pi_1^{\mathcal{G}}(p)$ to $\pi_1^{\overline{\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)}}(p)$, but this path runs entirely in $\mathcal{A}_k$ hence coincides with the path $\pi_1^{\mathcal{A}_k}(p)$ mentioned earlier.

By assumption, $[p]_G = [r]_G$ for some word $r \in \widetilde{B}^*$. The paths $\pi_1^{\mathcal{G}}(p)$ and $\pi_1^{\mathcal{G}}(r)$ have the same terminal vertex, namely $[p]_G = [r]_G$. The path $\pi_1^{\mathcal{G}}(r)$ is mapped by ψ_k onto the path $\pi_1^{\overline{\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)}}(r)$. But the B-component of 1 in $\overline{\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)}$ is the full B-coset $1\mathcal{H}_k[B]$ which is contained in $\mathsf{CE}(H_k, \mathcal{A}_k; B)$, hence the latter graph contains a path starting at 1 and being labelled r: $\pi_1^{\mathsf{CE}(H_k, \mathcal{A}_k; B)}(r)$, and that path actually runs inside $1\mathcal{H}_k[B]$. Since the paths $\pi_1^{\mathcal{G}}(r)$ and $\pi_1^{\mathcal{G}}(p)$ have the same terminal vertex, so have the paths

$$\pi_1^{1\mathcal{H}[B]}(r) = \pi_1^{\mathsf{CE}(H_k,\mathcal{A}_k;B)}(r) \text{ and } \pi_1^{\mathcal{A}_k}(p).$$

It follows that the terminal vertex v' of $\pi_1^{\mathcal{A}_k}(p)$ is in $\mathcal{A}_k \cap 1\mathcal{H}_k[B]$. But $\mathcal{A}_k \cap 1\mathcal{H}_k[B]$ is just the B-component of 1 in $\mathcal{A}_k$ which is a connected B-graph. Altogether, there exists a path $\pi\colon 1 \longrightarrow v'$ running in $\mathcal{A}_k \cap 1\mathcal{H}_k[B]$; let $q \in \widetilde{B}^*$ be the label of that path. By construction, $[q]_{H_k} = [r]_{H_k}$, hence $[q]_G = [r]_G$ since the expansion $H_k \twoheadleftarrow G$ is k-stable, and therefore also $[q]_G = [p]_G$. Finally, the canonical morphism $\varphi_k\colon \mathcal{H}_k \twoheadrightarrow \mathcal{X}_1$ maps $\pi = \pi_1^{\mathcal{A}_k \cap 1\mathcal{H}_k}(q)$ to a path in $\mathcal{A} \subseteq \mathcal{E}$ with initial vertex $u = \psi_k(1)$ and terminal vertex $v = \psi_k(v')$ and label q. Altogether, q forms a path $u \longrightarrow v$ in $\mathcal{E}$. $\square$

This proof sheds some light on the rôles that the components of $\mathcal{Z}_k$ play in the transition $H_k \rightsquigarrow G_{k+1}$. If there is a word p with $\mathrm{co}(p) = A$ and $|A| = k+1$ such that p forms a path $u \longrightarrow v$ in $\mathcal{E}$, and some letter $a \in A$ does not belong to the $H_k[A]$-content of p then the subgraph $\mathsf{CE}(H_k, \mathcal{A}_k; B)$ of $\overline{\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)}$ (for $\mathcal{A} = \langle A\rangle$ and $B = A \setminus \{a\}$) guarantees that the next group G_{k+1} avoids the relation $p = p_{a\to 1}$ (hence a does belong to the $G_{k+1}[A]$-content of p) unless there is a word $q \in \widetilde{B}^*$ such that $[p]_{H_k} = [q]_{H_k}$ and q forms a path $u \longrightarrow v$ in $\mathcal{E}$. From this point of view it would be sufficient to let $\mathcal{Z}_k$ be comprised of weak completions of all graphs $\mathsf{CE}(H_k, \mathcal{A}_k; B)$ of the mentioned kind. However, when attempting this approach the authors failed to prove k-stability of the expansion $H_k \twoheadleftarrow G_{k+1}$, and it is not clear whether or not k-stability can be achieved by this procedure. Hence, except for the graphs $\mathsf{CE}(H_k, \mathcal{A}_k; B)$ which appear as subgraphs of the full coset extensions $\mathsf{CE}(H_k, \mathcal{A}_k; \mathbb{P}_A)$ all the machinery used to set up the graph $\mathcal{Z}_k$ — (augmented) clusters, (augmented) full coset extensions, all of Section 4 — serves to achieve k-stability of the transition $H_k \rightsquigarrow G_{k+1}$.

If, in Lemma 5.5, $[p]_G = 1$ then necessarily $u = v$ since in this case the path $\pi_1^{\mathcal{G}}(p)$ is closed and the canonical morphism $\varphi_u\colon \mathcal{G} \twoheadrightarrow \mathcal{X}_1$ maps this path onto the path $\pi_u^{\mathcal{X}_1}(p) = \pi_u^{\mathcal{E}}(p)$. Iterated application of Lemma 5.5 leads to:

Corollary 5.6. *Let $p \in \widetilde{E}^*$ be a word which forms a path $u \longrightarrow v$ in $\mathcal{E}$; then there exists a word $q \in \widetilde{E}^*$ which uses only letters (i.e. edges) from the content $\mathrm{C}([p]_G)$ (and/or their inverses) such that $[p]_G = [q]_G$ and q forms a path $u \longrightarrow v$ in $\mathcal{E}$. If $\mathrm{C}([p]_G) = \varnothing$ then $u = v$ and q is the empty word. If $\mathrm{C}([p]_G) \neq \varnothing$ then the graph $\langle \mathrm{C}([p]_G)\rangle = \langle \mathrm{co}(q)\rangle$ is connected and contains the vertices u and v.*

5.4. **Final remark: pointlike conjecture for inverse monoids versus F-inverse cover problem.** What can we say about the gap between these two problems? As already mentioned, the truth of the pointlike conjecture for inverse monoids follows from Ash's result on inevitable labellings of graphs. The construction in Ash's paper is quite involved and the groups constructed there are not very well traceable. However, in [4] it has been shown that the expansion $Q^{\mathbf{Ab}_p}$ of an A-generated group Q witnesses the pointlike sets of the inverse monoid $M(Q)$ and therefore is able to verify the pointlike conjecture for inverse monoids. For any prime p, the so-called *universal p-expansion* $Q^{\mathbf{Ab}_p}$ of Q is the largest A-generated expansion R of Q with kernel of $R \twoheadrightarrow Q$ an elementary Abelian p-group. This expansion can be obtained by the construction in (2.1), except that the E-generated group G used there is replaced with the free E-generated Abelian group of exponent p (which is the $|E|$-fold direct product of cyclic groups of order p), in fact a very transparent group. Sufficient to verify the pointlike conjecture is an E-generated group which reflects the structure of the Cayley graph $\mathcal{Q}$ of Q in a very weak sense: the graph spanned by the content of a word over $\widetilde{E}$ which forms a path $u \longrightarrow v$ requires only a *connected component* containing u and v. From this point of view, it seems to be justified to say that the gap between the pointlike problem for inverse monoids and the F-inverse cover problem is huge, indeed.

As already mentioned, Henckell and Rhodes considered Problem 1.1 as a “stronger form” of the pointlike conjecture for inverse monoids. On the other hand, in the last sentence of their paper they wrote: “We do not necessarily believe [the F-inverse cover problem] has an affirmative answer.” So, in contrast to what is often reported, Henckell and Rhodes did not really conjecture that every finite inverse monoid does admit a finite F-inverse cover, but rather seem to have been undecided about this question. In fact, they seem to have had some feeling that the F-inverse cover problem might be hard.

References

[1] J. Almeida and M. Delgado, Sur certains systèmes d'équations avec contraintes dans un groupe libre, *Portugal. Math.* **56** (1999), no. 4, 409–417.

[2] J. Almeida and M. Delgado, Sur certains systèmes d'équations avec contraintes dans un groupe libre — addenda, *Portugal. Math.* **58** (2001), no. 4, 379–387.

[3] C. J. Ash, Inevitable graphs: a proof of the type II conjecture and some related decision procedures, *Internat. J. Algebra Comput.* **1** (1991), no. 1, 127–146.

[4] K. Auinger and B. Steinberg, The geometry of profinite graphs with applications to free groups and finite monoids, *Trans. Amer. Math. Soc.* **356** (2004), no. 2, 805–851.

[5] K. Auinger and M. B. Szendrei, On F-inverse covers of inverse monoids, *J. Pure Appl. Algebra* **204** (2006), 493–506.

[6] K. Auinger and A. Bors, Formations of finite groups with the M. Hall property, *J. Comb. Algebra* **3** (2019), no. 2, 113–152.

[7] K. Auinger, G. Kudryavtseva and M. B. Szendrei, F-inverse monoids as algebraic structures in enriched signature, *Indiana Univ. Math. J.* **70** (2021), no. 5, 2107–2131.

[8] B. Billhardt, B. Singha, W. Sommanee, P. Thamkaew, J. Tiammee, A note on dual prehomomorphisms from a group into the Margolis–Meakin expansion of a group, *Semigroup Forum* **101** (2020), no. 3, 534–546.

[9] J. Bitterlich, Finite F-inverse covers do exist, preprint (2018), arXiv:1808.05161.

[10] J. Bitterlich, Investigations into the Universal Algebra of Hypergraph Coverings and Applications. Dissertation, Technische Universität Darmstadt, 2019.

[11] K. Henckell and J. Rhodes, The theorem of Knast, the $PG = BG$ and type-II conjectures, Monoids and semigroups with applications (Berkeley, CA, 1989), 453–463, World Sci. Publ., River Edge, NJ, 1991.

[12] B. Herwig and D. Lascar, Extending partial automorphisms and the profinite topology on free groups, *Trans. Amer. Math. Soc.* **352** (2000), no. 5, 1985–2021.

[13] I. Kapovich and A. Myasnikov, Stallings foldings and subgroups of free groups, *J. Algebra* **248** (2002), no. 2, 608–668.

[14] M. V. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries*, World Scientific, River Edge, NJ, 1998.

[15] S. W. Margolis and J. C. Meakin, E-unitary inverse monoids and the Cayley graph of a group presentation, *J. Pure Appl. Algebra* **58** (1989), 45–76.

[16] M. Otto, Highly acyclic groups, hypergraph covers, and the guarded fragment, *J. ACM* **59** (2012), no. 1, Art. 5, 40 pp.

[17] M. Otto, Groupoids, hypergraphs, and symmetries in finite models. (English summary) 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2013), 53–62, IEEE Computer Soc., Los Alamitos, CA, 2013.

[18] M. Otto, Finite groupoids, finite coverings and symmetries in finite structures, preprint (2014) arXiv:1404.4599.

[19] M. Otto, Acyclicity in finite groups and groupoids, preprint (2021), arXiv:1806.08664v4.

[20] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.

[21] L. Ribes and P. Zalesskiĭ, On the profinite topology on a free group, *Bull. London Math. Soc.* **25** (1993), no. 1, 37–43.

[22] J. R. Stallings, Topology of finite graphs, *Invent. Math.* **71** (1983), no. 3, 551–565.

[23] N. Szakács, On the graph condition regarding the F-inverse cover problem, *Semigroup Forum* **92** (2016) no. 3, 551–558.

[24] N. Szakács and M. B. Szendrei, On F-inverse covers of finite-above inverse monoids, *J. Algebra* **452** (2016), 42–65.

[25] M. B. Szendrei, Some open problems in the structure theory of regular semigroups, *Semigroup Forum* **64** (2002), no. 2, 213–223.

[26] M. B. Szendrei, Problems on finite regular semigroups, Proceedings of the International Symposium on Semigroups and Applications, 37–45, Dept. Math. Univ. Kerala, Thiruvananthapuram, 2007.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria
E-mail address: `karl.auinger@univie.ac.at`

Department of Mathematics, Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64289 Darmstadt, Germany
E-mail address: `otto@mathematik.tu-darmstadt.de`