# Invariances, Games, Models
## and (Finite) Model Theory in Tractable Fragments of FO

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# domain specific modelling

## semantic domains:

**class of structures**

| | |
|---|---|
| all | classical model theory |
| finite | finite model theory |
| connected | |
| acyclic | |
| . . . | |

**notion of equivalence**

| | |
|---|---|
| isomorphism | algebra |
| elementary equiv. | classical model theory |
| bisimulation equiv. | |
| . . . | |

$$\left.\begin{array}{r}\textbf{structures}\\ \text{modulo}\\ \textbf{equivalence}\end{array}\right\}\quad \text{to model}\quad \left\{\begin{array}{l}\textbf{objects}\\ \text{at the right level of}\\ \textbf{abstraction}\end{array}\right.$$

## choice of logic

governed by
- **expressiveness requirements**
- **algorithmic requirements**
- **invariance requirements**

choice of logic subordinate to semantic domain

cf. fundamentals of geometry / F. Klein

## samples of domains with tractable logics

**modal**

| | |
|---|---|
| **bisimulation eq.** | $\sim$ |
| Milner, van Benthem | |

$\Box_E \varphi(x) \equiv \forall y \big( Exy \to \varphi(y) \big)$ **ML**

**2-variable**

| | |
|---|---|
| **2-pebble game eq.** | $\simeq^2$ |
| Barwise, Poizat, Immerman | |

two variables $x$,$y$ (free or bound) **FO$^2$**

**guarded**

| | |
|---|---|
| **guarded bisim. eq.** | $\sim^{\mathsf{G}}$ |
| Andréka, van Benthem, Németi | |

$\forall \boldsymbol{y} \big( \alpha(\boldsymbol{x}, \boldsymbol{y}) \ \to \varphi(\boldsymbol{x}, \boldsymbol{y}) \big)$ **GF**

guard atom: free($\varphi$) $\subseteq$ var($\alpha$)

## semantics – games – syntax

| | |
|---|---|
| modal equiv. | bisimulation game |
| $k$-variable equiv. | $k$-pebble game |
| guarded equiv. | guarded bisimulation game |

$\longrightarrow$ **Ehrenfeucht-Fraïssé theorems**

Barwise, van Benthem, Poizat, Immerman, . . .

$\longrightarrow$ **characterisation theorems**

# semantics – games – syntax

**invariances and characterisation theorems (classically)**
**effective syntax for undecidable invariances**

## FO-fragments

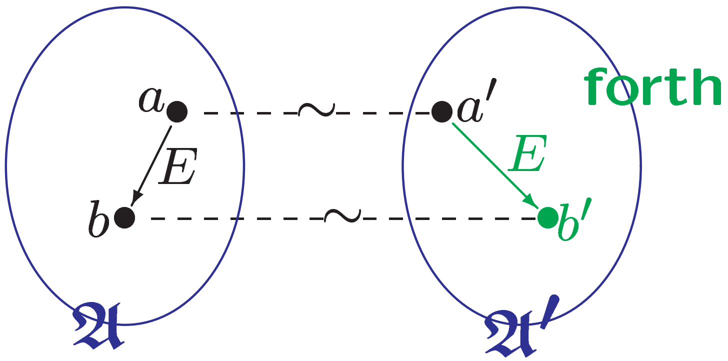
| | | | |
|---|---|---|---|
| $\mathrm{FO}/\!\sim$ | $\equiv$ | ML | van Benthem |
| $\mathrm{FO}/\!\simeq^k$ | $\equiv$ | $\mathrm{FO}^k$ | Barwise, van Benthem |
| $\mathrm{FO}/\!\sim_{\mathbf{g}}$ | $\equiv$ | GF | Andréka, van Benthem, Németi |

## tractable SO-fragments

| | | | |
|---|---|---|---|
| $\mathrm{MSO}/\!\sim$ | $\equiv$ | $\mathrm{L}_\mu$ | Janin,Walukiewicz'96 |
| $\mathrm{GSO}/\!\sim_{\mathbf{g}}$ | $\equiv$ | $\mu$GF | Grädel,Hirsch,O'00 |

## **bisimulations** between coloured graphs $\mathfrak{A} = (A, P, \ldots, E, \ldots)$

**back & forth along edges**




$\boldsymbol{Z} : \mathfrak{A} \sim \mathfrak{A}'$

- $\emptyset \neq Z \subseteq A \times A'$
- respects unary predicates
- back & forth along edges

**Ehrenfeucht-Fraïssé for modal quantification**

$$\Box_E \varphi(x) \equiv \forall y \big( \, Exy \to \varphi(y) \, \big)$$

relativised along edges

**variations: two-way and global bisimulations**

**ML** **basic modal logic** atomic propositions (unary predicates)
closure under Boole, and
modal quantification $\Box, \Diamond$

**bisimulation invariance of $\varphi \in$ ML**

$$\mathfrak{A}, a \sim \mathfrak{A}', a' \;\Rightarrow\; \mathfrak{A}, a \models \varphi \;\Leftrightarrow\; \mathfrak{A}', a' \models \varphi$$

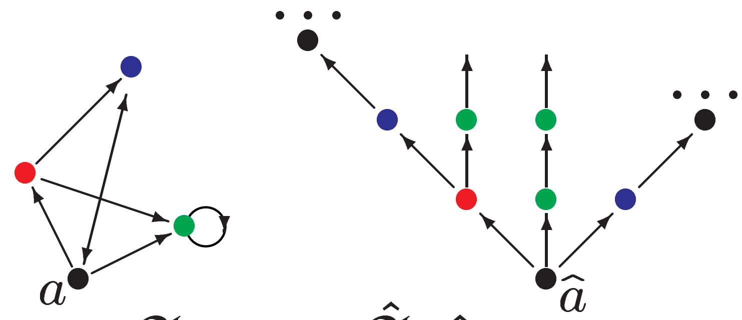
**van Benthem's characterisation theorem**

$$\mathrm{ML} \equiv \mathrm{FO}/\!\sim$$

$\varphi \in$ FO : $\varphi$ **bisimulation invariant**
$\Leftrightarrow$ $\varphi$ **expressible in ML**

## tree model property

**tree unravellings**: $\longrightarrow$ bisimilar companions that are trees
bisimilar companions
built from tree of paths



$$\mathfrak{A}, a \;\sim\; \hat{\mathfrak{A}}, \hat{a}$$

$\Rightarrow$ **reductions to model theory of trees**
for any bisimulation invariant (i.e. modal) logic

$\left.\begin{array}{l}\textbf{Rabin's Theorem}\\ \textbf{tree automata}\end{array}\right\}$ for decidability issues

## characterisation theorems $\mathcal{L} \equiv \mathrm{FO}/\!\approx$ — classical vs. finite model theory

$\varphi \in \mathrm{FO}$:

| | | |
|---|---|---|
| $\approx$-invariant over finite models | $\Leftrightarrow$ | expressible in $\mathcal{L}$ over finite models |

### independent questions in general

- $\mathrm{FO}^2 \equiv \mathrm{FO}/\!\simeq^2$ not valid in fmt — $\varphi(<)$: "linear ordering"
- $\sim$-invariance in finite models $\not\Rightarrow$ $\sim$-invariance — $\varphi(<) \wedge \forall x \exists y\, (x < y)$
- fmp for ML no good
- classical proofs don't go through

## modal characterisation theorems do go through!

$\mathrm{FO}/{\sim} \equiv \mathrm{ML}$ classically and in fmt

van Benthem Rosen'97

new tight bounds: $\sim$-invariant $\varphi \in \mathrm{FO}$

expressible in ML at depth $2^{\mathrm{qr}(\varphi)}$

elementary Ehrenfeucht-Fraïssé argument!

new ramifications: $\mathrm{FO}/{\sim_\forall} \equiv \mathrm{ML} + \forall$

$\mathrm{FO}/{\sim_\forall^{\leftrightarrow}} \equiv \mathrm{ML} + \forall + \Box^-$

$\mathrm{FO}/{\sim_\mathrm{g}} \equiv \mathrm{GF}$ (arity $2$ structures)

classically, without arity restrictions:

Andréka,van Benthem,Németi

## E-F games and characterisation theorems

use finite approximations $\approx^\ell$ for full equivalence $\approx$

crux of characterisation thms, classical & fmt:

$$\boxed{\varphi \approx\text{-invariant} \quad \Rightarrow \quad \varphi \approx^\ell\text{-invariant for some } \ell}$$

classically

$$\begin{array}{ccc} \mathfrak{A} & \approx^\ell & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \\ & \text{all } \ell & \end{array}$$

upgrade via
compactness
$\omega$-saturation

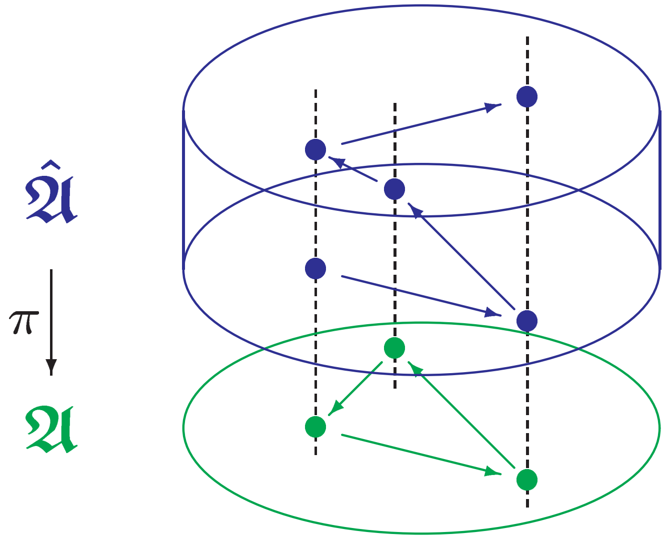
$$\begin{array}{ccc} \mathfrak{A} & \approx & \mathfrak{A}' \\ \models \varphi & & \models \neg\varphi \end{array}$$

## nice models up to $\sim$ : acyclic OR finite

cannot have acyclic finite bisimilar companions for cyclic models

## from unravellings to bisimilar covers

homomorphism $\pi\colon \hat{\mathfrak{A}} \to \mathfrak{A}$
whose graph induces a
two-way global bisimulation




**two-way unravellings are (infinite) acyclic covers**

## finite locally acyclic covers

### theorem

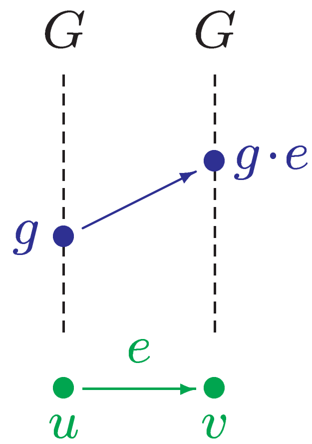
for every $k \in \mathbb{N}$ and finite transition system $\mathfrak{A}$
there is a **finite bisimilar cover $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$**
where **$\hat{\mathfrak{A}}$ $\mathrm{k}$-locally acyclic**
**$\pi$ degree-preserving**

notes:
- short *directed* cycles are easily avoided
- here use group operations on underlying tournament
- polynomial construction available for fixed $k$

## locally acyclic covers via group action

geometric intuition: fibre bundle over $A$

to cover $\mathfrak{A} = (A, E)$ by $\hat{\mathfrak{A}} = (\hat{A}, \hat{E})$




**let $E \hookrightarrow G$ operate on group $G$**

$\boldsymbol{\hat{A}}$**:** $A \times G$

$\boldsymbol{\hat{E}}$**:** $((u,g),(v,h)) \in \hat{E} \Leftrightarrow$ $(u,v) = e \in E$ and $h = g \cdot e$

$\boldsymbol{G}$ generated by $\boldsymbol{H}$ with large girth

g$(\hat{\mathfrak{A}}) \geqslant$ g(Cayley$(G, H)$)

polynomial construction for fixed $k$

Margulis, Imrich

## why local control suffices

**Gaifman graph** induces distance / notion of locality
edge $(a, a')$ iff $a$ and $a'$ occur together in any relation

## Gaifman's locality theorem

FO can only express Gaifman local properties

with ramifications for formulae invariant under disjoint sums

for simple proof of van Benthem–Rosen, direct via E-F games

**Lemma**

$\varphi \in$ FO invariant under disjoint sums $\Rightarrow$ $\varphi$ $\ell$-local for $\ell \leqslant 2^{\mathrm{qr}(\varphi)}$

**crux of characterisation thms, classical & fmt:**

$$\boxed{\varphi \approx\text{-invariant} \;\Rightarrow\; \varphi \approx^{\ell}\text{-invariant for some } \ell}$$

**classically**

$$\underset{\models\varphi}{\mathfrak{A}} \quad \underset{\text{all } \ell}{\approx^{\ell}} \quad \underset{\models\neg\varphi}{\mathfrak{A}'}$$

upgrade via
**compactness**
**$\omega$-saturation**

$$\underset{\models\varphi}{\mathfrak{A}} \quad \approx \quad \underset{\models\neg\varphi}{\mathfrak{A}'}$$

**alternative, constructive proof**

$$\underset{\models\varphi}{\mathfrak{A}} \quad \underset{\ell=\ell(\varphi)}{\approx^{\ell}} \quad \underset{\models\neg\varphi}{\mathfrak{A}'}$$

**$\ell$-acyclic**
**$\approx$ covers**
**(finite)**

$$\underset{\models\varphi}{\mathfrak{A}} \quad \underset{q=\mathrm{qr}(\varphi)}{\equiv_{q}^{(\ell)}} \quad \underset{\models\neg\varphi}{\mathfrak{A}'}$$

## the methodological shift

trading bisimilarity $\approx^\ell$, $\approx$ vs. elementary equivalence $\equiv$, $\equiv_q^{(\ell)}$

| | | |
|---|---|---|
| **classically**: | **preserving** $\equiv$ <br> **upgrade** $(\approx^\ell)$ **to** $\approx$ | **compactness** |
| **fmt**: | **preserving** $\approx$ <br> **upgrade** $\approx^\ell$ **to** $\equiv_q^{(\ell)}$ | **locally acyclic covers** |

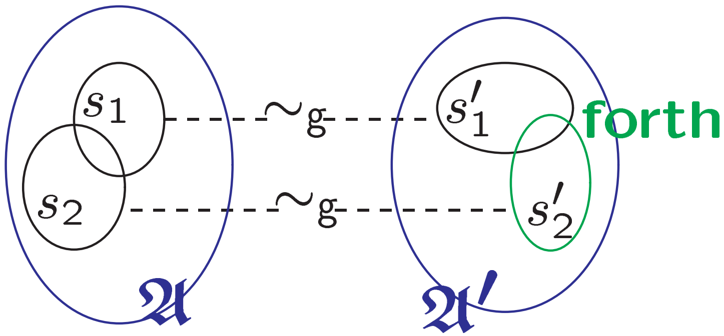
- fmt proof idea applies similarly in classical context
- covers several variations: $\sim_\forall$, $\sim_{\overline{\forall}}$, $\sim_{\mathbf{g}}$

## guarded bisimulation and guarded logics

over arbitrary relational structures

**guarded subsets**: visible patches,
covered by relational hyper-edges

**back & forth over hyper-edges**



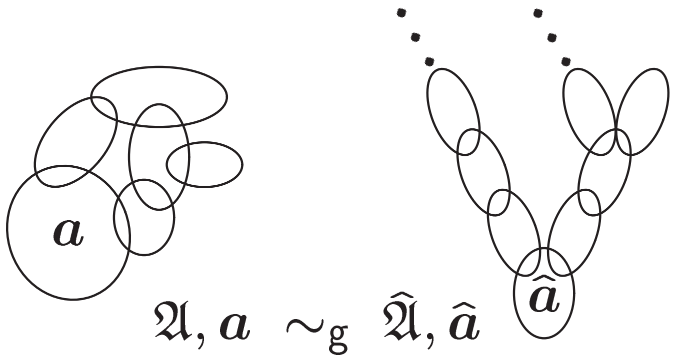

$$\boldsymbol{Z : \mathfrak{A} \sim_{\mathrm{g}} \mathfrak{A}'}$$

- $\emptyset \neq Z \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{A}')$
- back & forth w.r.t. guarded subsets compatible in overlaps

## generalised tree model property

for any $\sim_{\mathrm{g}}$ invariant (i.e. guarded) logic

**guarded tree unravellings**: guarded bisimilar companions built from tree of guarded paths




$\mathfrak{A}, \boldsymbol{a} \sim_{\mathrm{g}} \widehat{\mathfrak{A}}, \widehat{\boldsymbol{a}}$

**guarded tree decomposable**
of **bounded tree width**

$\Rightarrow$ **reductions to model theory of trees**
via interpretations in trees

## the modal and guarded fragments

| | **modal: ML** | **guarded: GF** |
|---|---|---|
| | $\forall y\big(Rxy \to \varphi(y)\big)$<br>relativised to edges | $\forall \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \to \varphi(\boldsymbol{x},\boldsymbol{y})\big)$<br>relativised to guarded subsets |
| **Ehrenfeucht-Fraïssé:** | | |
| | bisimulation $\sim$ | guarded bisimulation $\sim_{\mathrm{g}}$ |
| **unravellings:** | | |
| | tree models | guarded tree decomp.<br>bounded tree width |
| **characterisations:** | | |
| | $\mathsf{ML} \equiv \mathsf{FO}/\!\sim$ | $\mathsf{GF} \equiv \mathsf{FO}/\!\sim_{\mathbf{g}}$ |

## finite guarded covers

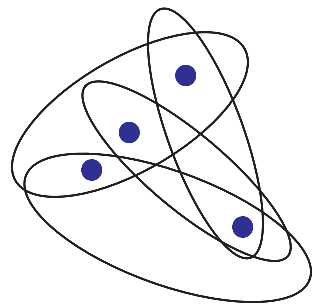
**finite locally acyclic guarded covers for transition systems**
by extension of above, also proving $\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$ in this case
**open in general**

guarded unravellings $\rightarrow$ **acyclic hypergraphs**

| **conformal** | and | **chordal** |
|---|---|---|
| no bad cliques | | no bad cycles |

**bad cliques: unguarded Gaifman cliques**
**bad cycles: chordless Gaifman cycles**

unguarded Gaifman cliques
are broken up in unravellings



allowed as visible
patches in CGF

## finite conformal covers

combinatorial construction for hypergraphs

**theorem** (Hodkinson, O'01)

for every finite relational $\mathfrak{A}$
there is a **guarded cover** $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$
by a **finite conformal** $\hat{\mathfrak{A}}$

geometric intuition: local "generic" cross-sections in fibre bundle over $A$
canonical and symmetry preserving, $\mathrm{Aut}(\mathfrak{A}) \hookrightarrow \mathrm{Aut}(\hat{\mathfrak{A}})$

**applications**

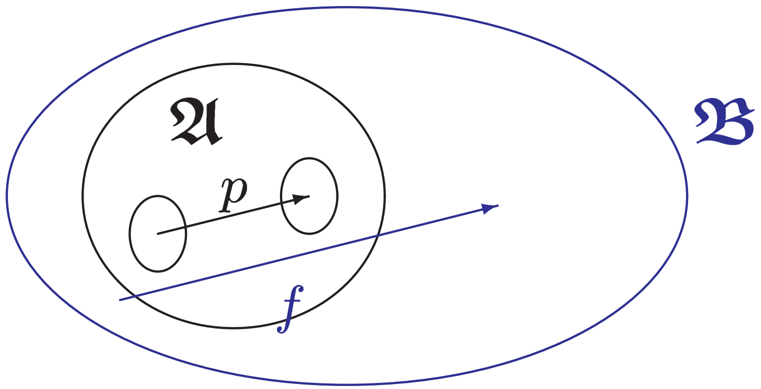
- finite model theory of guarded logics
- **extension properties for partial maps**

extension properties for partial maps

## Hrushovski/Herwig/Lascar extension thms **EPPA**

profinite topology of free groups (Hall; Ribes, Zalesskii)
automorphism groups of countable structures
(Truss, Hodges, Hodkinson, Lascar, Shelah)

for finite relational $\mathfrak{A}$ find finite extension $\mathfrak{B}$
extending every $\boldsymbol{p} \in \mathbf{Part}(\mathfrak{A})$ to $\boldsymbol{f} \in \mathbf{Aut}(\mathfrak{B})$




call extension **#-faithful**
if $\mathfrak{B}$ realises no "new" #

## **clique faithful extension thm** Hodkinson, O'01

EPPA extensions without "new" Gaifman cliques

uses finite conformal covers on top of H/H/L construction

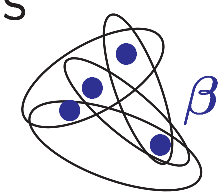
## **corollary** (Herwig/Lascar)

EPPA for the class of finite $K_n$-free graphs

elementary proof when combined with Lascar's EPPA for graphs

## **corollary** (Hodkinson)

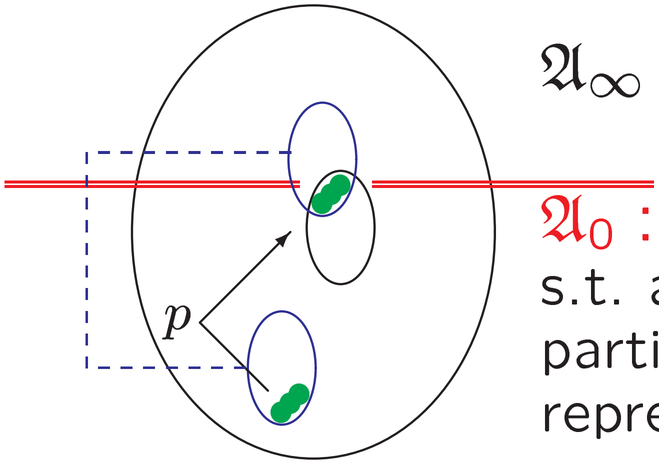
finite model property for CGF – with Gaifman cliques as visible patches



$\beta(y)$

# extension properties and finite models

## finite closure of partial model (after Skolemisation)




$\mathfrak{A}_\infty \models \varphi$

$\mathfrak{A}_0$ : a large finite segment s.t. all missing witnesses have partially isomorphic representatives in $\mathfrak{A}_0$

$\varphi$ holds in #-faithful EPPA extension of $\mathfrak{A}_0$

- guarded-set faithful for GF
- clique faithful for CGF