Logics – Invariances – Games

Domain Specific Algorithmic Model Theory

A Model Theoretic Perspective on Logic in Computer Science

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mathematical modelling in logic and model theory for CS

relevant entities	**as structures**
databases	**relational structures**
words, streams	**monadic structures with order**
processes (temporal)	**graphs** **transition systems**
semantic dependencies (epistemic)	**graphs** **Kripke structures**

possibly restrict to finite structures $\rightarrow$ **FMT**

when two structures encode the same thing

isomorphism may be too fine
e.g., **process behaviour** $\approx$ **bisimulation type**

the modelling task

the semantic domain

determine $\left\{\begin{array}{l}\text{basic } \textbf{class of structures} \\ \text{and } \textbf{notion of equivalence}\end{array}\right.$

to match $\left\{\begin{array}{l}\text{the right } \textbf{objects} \\ \text{at the right } \textbf{level of abstraction}\end{array}\right.$

the logic

chosen to $\left\{\begin{array}{l}\text{meet } \textbf{expressiveness requirements} \\ \text{respect } \textbf{prescribed equivalence}\end{array}\right.$

choice of logic subordinate to
choice of semantic domain

examples of semantic domains

modal	—	bisimulation eq.	$\sim$
		Milner,van Benthem	
2-**variable**	—	2-pebble game eq.	$\simeq^2$
		Barwise,Poizat,Immerman	
guarded	—	guarded bisim. eq.	$\sim^{\mathsf{G}}$
		Andreka,van Benthem,Nemeti	

and many others: (guarded) k-variable,
k-var. with counting,
topological, affine, . . .

some patterns of logics

modal logic, ML $\sim$

with modal quantification

$$\begin{aligned}\Diamond\varphi(x) &\equiv \exists y(Exy \wedge \varphi(y)) \equiv \exists y.Exy\,\varphi(y)\\ \Box\varphi(x) &\equiv \forall y(Exy \to \varphi(y)) \equiv \forall y.Exy\,\varphi(y)\end{aligned}$$

2-variable first-order logic, FO^2 $\simeq^2$

with variables x,y; relational, with $=$

guarded first-order logic, GF $\sim^{\mathsf{G}}$

with guarded quantification

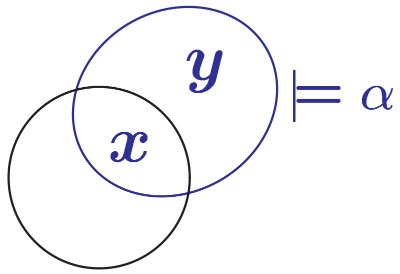
$$\begin{aligned}\exists \boldsymbol{y}\Big(\alpha(\boldsymbol{x},\boldsymbol{y}) \wedge \varphi(\boldsymbol{x},\boldsymbol{y})\Big) &\equiv \exists \boldsymbol{y}.\alpha(\boldsymbol{x},\boldsymbol{y})\,\varphi(\boldsymbol{x},\boldsymbol{y})\\ \forall \boldsymbol{y}\Big(\alpha(\boldsymbol{x},\boldsymbol{y}) \to \varphi(\boldsymbol{x},\boldsymbol{y})\Big) &\equiv \forall \boldsymbol{y}.\alpha(\boldsymbol{x},\boldsymbol{y})\,\varphi(\boldsymbol{x},\boldsymbol{y})\end{aligned}$$

for **guards** α: atoms covering free(φ)

infinitary variants **ML$_\infty$, FO$^2_\infty$, GF$_\infty$**
with infinite $\bigvee$, $\bigwedge$

ML $\longrightarrow$ GF

- generalises quantification pattern
- no arity restrictions
- comprises $=$, inverses, universal modalities

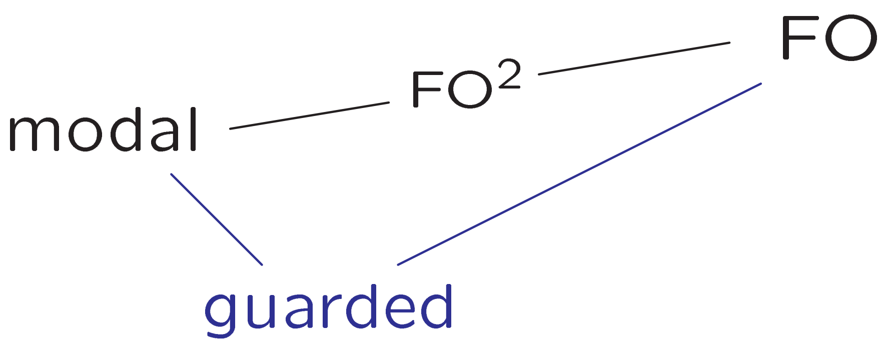


$\exists y.\alpha(x,y)$___

modal domain $\longrightarrow$ guarded domain

- bisimulation $\rightarrow$ guarded bisimulation
- states $\rightarrow$ guarded patches locality
- transitions $\rightarrow$ overlaps adjacency

GF shares FMP, decidability, . . .



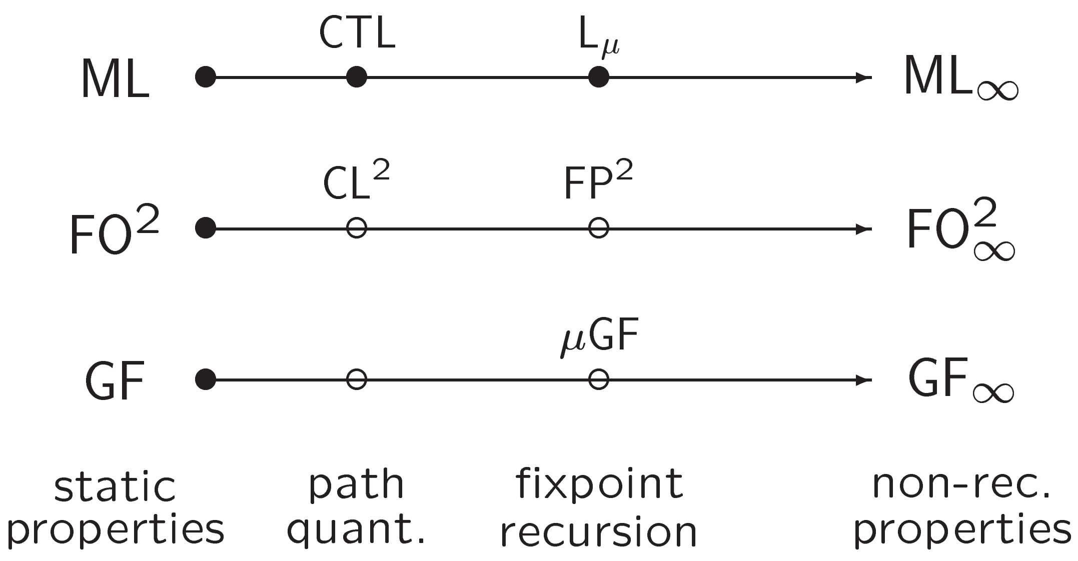
semantics – games – syntax

modal equiv.	bisimulation game
k-variable equiv.	k-pebble game
guarded equiv.	guarded bisimulation game

$\rightarrow$ **Ehrenfeucht-Fraïssé style characterisation thms**

Barwise, van Benthem, Poizat, Immerman, . . .

domains come with spectra of logics



semantic completeness – **logic and invariance**

characterisation thms

FO-fragments

$\mathrm{FO}/\sim$	$\equiv$	ML	van Benthem; Rosen (FMT)
$\mathrm{FO}/\simeq^k$	$\equiv$	FO^k	Barwise,van Benthem
$\mathrm{FO}/\sim^{\mathrm{G}}$	$\equiv$	GF	Andreka,van Benthem,Nemeti

tractable SO-fragments

$\mathrm{MSO}/\sim$	$\equiv$	L_μ	Janin,Walukiewicz
$\mathrm{GSO}/\sim^{\mathrm{G}}$	$\equiv$	$\mu\mathrm{GF}$	Grädel,Hirsch,Otto

$\mathrm{GSO} = \mathrm{SO} \restriction \text{guarded}$: natural counterpart to MSO in the guarded domain

translations **guarded** $\longleftrightarrow$ **modal**

proof of the μGF characterisation thm

guarded		**modal**
$\mathfrak{A}$	$\xrightarrow[\text{tree repr. of unravelling}]{}$	$\boldsymbol{T}(\hat{\mathfrak{A}})$
$\mathfrak{A}(\boldsymbol{T})$	$\xleftarrow[\text{induced structure}]{}$	$\boldsymbol{T}$
$\sim^{\mathsf{G}}$	$\longleftrightarrow$	$\sim$
GSO	$\longrightarrow$	MSO
μGF	$\longleftarrow$	L_μ

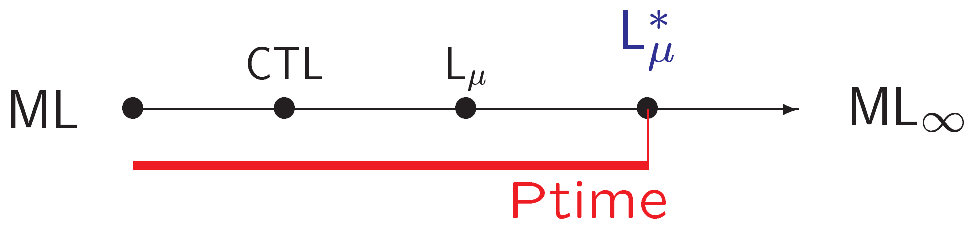
back-and-forth in more than one sense

semantic completeness – logic and complexity

capturing levels of computational complexity in machine independent fashion

capturing results for $\begin{cases} \text{Ptime}/\sim & \text{O'97} \\ \text{Ptime}/\simeq^2 & \text{O'95} \end{cases}$

e.g.



L_μ^* **poly-adic μ-calculus**
with fixed-points in all arities
respecting $\sim$ componentwise

natural extension of L_μ, undecidable for SAT

capturing Ptime via game invariants

capturing Ptime via **game invariants**

$\mathfrak{A}$	$\longmapsto$	$\mathrm{I}(\mathfrak{A})$
		ordered quotient w.r.t. $\approx$-game
		a complete invariant for $\mathfrak{A}/\approx$

$$\longrightarrow \quad \mathbf{Ptime}/\approx\; \equiv \; \mathbf{Ptime} \circ \mathbf{I} \; \equiv \; \mathbf{FP}(\mathbf{I})$$

e.g., in **modal domain**:

$$\begin{aligned} \mathrm{Ptime}/\sim &= \text{Ptime on quotients w.r.t. } \sim \\ &= \text{FP on quotients w.r.t. } \sim \\ &= \mathrm{FP}/\sim \\ &= \mathrm{L}^*_\mu \end{aligned}$$

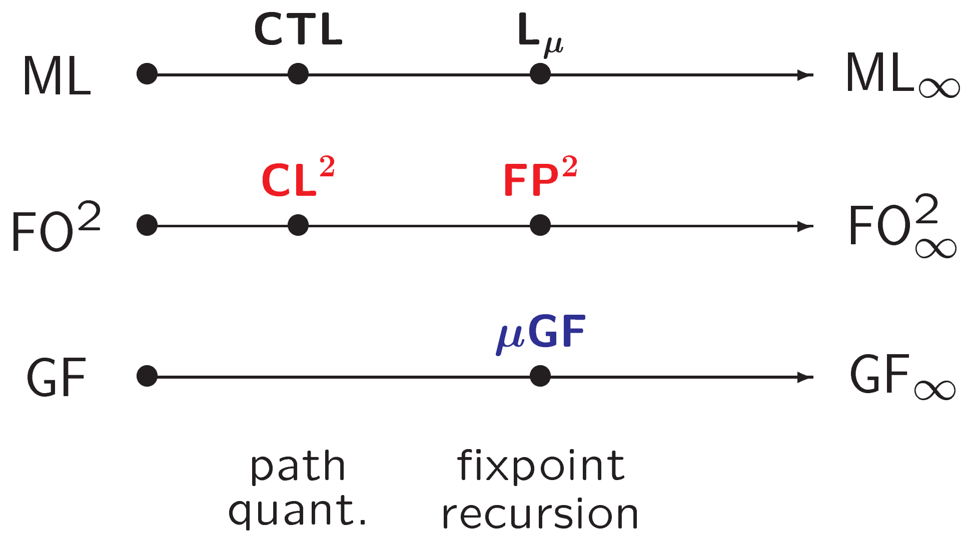
$\rightarrow$ capturing results for $\simeq^2$ / $\simeq^2_{\mathrm{C}}$
$\rightarrow$ limitations for $k > 2$ variables (Grohe)

OPEN: extension to guarded domain

satisfiability: SAT and FINSAT

explore borderline of decidability in related domains

understand modal robustness(Vardi)

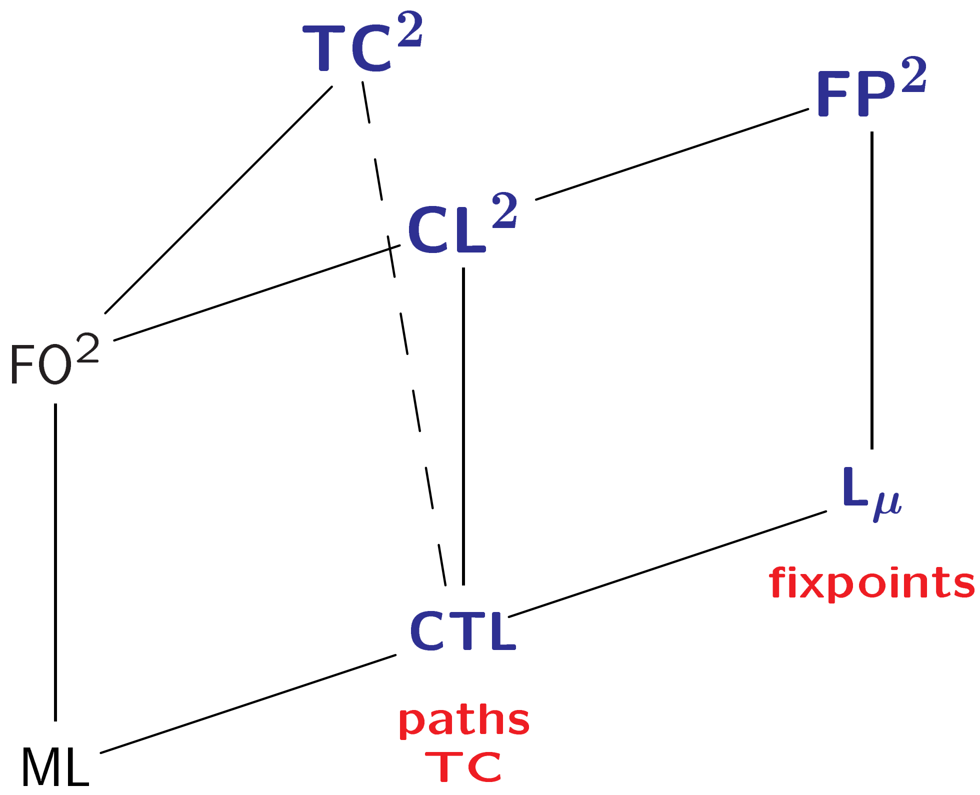


modal robustness
not inherited in 2-var. domain
but in guarded domain

CL^2 undecidable O'97→GOR'97→O'01

μGF decidable Grädel,Walukiewicz'99

two-variable candidates (of a sort)



$\mathbf{FP^2}$: two-variable fixed-point logic

monadic least and greatest fixed points

$\mathbf{TC^2}$: two-variable transitive closure logic

$*$-operation on actions
transition based reachability

$\mathbf{CL^2}$: two-variable path closure logic

state based universal/existential reachability

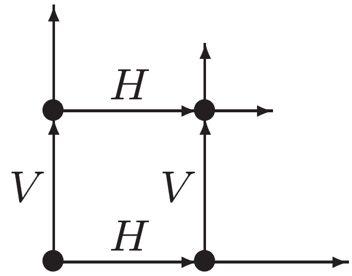
undecidability via dominoes

domino system $\mathcal{D} = (D, R_h, R_v)$

D finite set of tile types

$R_{h/v} \subseteq D^2$ hor./vert. adjacency

$\mathcal{D}$-tiling $\big(\mathbb{N}\times\mathbb{N}, H, V\big) \xrightarrow{\text{hom}} \mathcal{D}$



domino problem:

the associated constraint satisfaction problem

solutions: constrained D-colourings of $\mathbb{N} \times \mathbb{N}$

constraints in FO^2: $\psi_{\mathcal{D}}$
not in FO^2: grid conditions

Berger'66,Gurevich,Koryakov'72

(D1) $\mathcal{D}$-tilings
(D2) periodic (finite) $\mathcal{D}$-tilings

undecidable
recursively inseparable

undecidability results
varying domains versus **varying logics**

over suitable classes $\mathcal{C}$:

simultaneous reduction

from (D1)/(D2) to $\mathbf{SAT/FINSAT(FO^2)}$ **over** $\mathcal{C}$

and characterisation of $\mathcal{C}$ in suitable L

classes $\mathcal{C}$	**logics** L
$\mathfrak{A} = \left(A, E_1, \dots, E_k, \bar{E}, \bar{P}\right)$ where the $E_1, \dots, E_k$ are	
equivalence relations	FP^2, TC^2
well-founded	FP^2, CL^2
well-orderings	FP^2, CL^2
transitive	TC^2

conservative reduction classes

contrast modal frame conditions

recursion mechanisms —
the boundedness problem

eliminability of fixed-point recursion, e.g., in

$$\mathrm{ML} \;\bullet\!\!\longrightarrow\!\!\overset{\mathrm{L}_\mu}{\bullet}\!\!\longrightarrow\; \mathrm{ML}_\infty$$

inductive generation of least fixed-points

$\varphi(X, x)$, positive in X

$\mu_X \varphi = \bigcup_\alpha \varphi^\alpha$	**least fixed point**
$\varphi^\alpha = \left\{ \boldsymbol{x} : \varphi\left[\bigcup_{\beta<\alpha} \varphi^\beta, \boldsymbol{x} \right] \right\}$	**stages**
$\gamma_{\mathfrak{A}} = \mathrm{min}_\alpha(\varphi^\alpha[\mathfrak{A}] = \varphi^{\alpha+1}[\mathfrak{A}])$	**closure ordinal**

$\varphi(X, \boldsymbol{x})$ **bounded** if $\gamma_{\mathfrak{A}}$ uniformly bounded

Barwise-Moschovakis-Theorem

$$\varphi(X, \boldsymbol{x}) \in \mathrm{FO\ bdd} \quad \Leftrightarrow \quad \mu_X \varphi \ \mathrm{FO\text{-}definable}$$

specialises to natural fragments

boundedness as a decision problem

BD(L)	
given:	$\varphi(X, \boldsymbol{x}) \in$ L, positive in X
decide:	is φ bounded?

SAT(L) $\leqslant$ BD(L) for $\wedge$-closed L with non-trivial BD(L)

classical results in CS concern BD(Datalog)
$\rightarrow$ query optimization

Cosmadakis, Gaifman, Hillebrand
Kanellakis, Mairson, Sagiv, Vardi 87/90

not candidates like $\exists$-FO, $\forall$-FO, FO^2, ML, GF

domain specific results:

modal	**decidable**	O'99
2 variables	**undecidable**	Kolaitis, O'98
guarded	**OPEN**	

undecidability in 2-variable domain

$\mathrm{BD}(\mathbf{FO}^2)$ undecidable

monadic, no $\neg$, $=$; w.r.t. universal recursion

proved via reduction of domino problem
based on König's lemma:

$\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$ $\quad\Leftrightarrow\quad$ $\mathcal{D}$ tiles arbitrarily large $n\times n$
a combinatorial boundedness issue

decidability in modal domain

in modal domain, even

recursion-free μ-calculus
given: $\psi \in \mathrm{L}_\mu$
decide: ex. $\psi_0 \in \mathrm{ML}$, $\psi \equiv \psi_0$?

"spurious recursion" **decidable in Exptime**

proved via bisimulation-invariance
tree models
tree automata / S2S

$\rightarrow$ BD(ML) result with modal B-M-Thm

summary

domain specific algorithmic model theory

modelling

games
invariances

technical tools — **methodological insights**

e.g., robustness of modal domain

bisimulation invariance

tree model property

- $\rightarrow$ generalizes to guarded domain
- $\rightarrow$ automata theory/S2S

e.g., modal $\longleftrightarrow$ guarded

explanatory power and technical transfer

potential for extension of **modal technology**
via development of **guarded model theory**