Bisimulation Invariance over Transitive Frames

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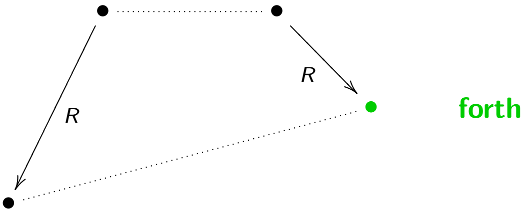
joint work with Anuj Dawar

bisimulation

$\sim$	bisimulation equivalence infinitary back&forth game
$\sim^{\ell}$	finite approximation to depth ℓ ℓ-round back&forth game

the game equivalence
modal Ehrenfeucht–Fraïssé



expressive completeness results for modal logics

van Benthem–Rosen

$\mathbf{FO}/{\sim} \equiv \mathbf{ML}$ over the class of $\left\{ \begin{array}{l} \text{all Kripke structures} \\ \text{all finite Kripke structures} \end{array} \right.$

Hafer–Thomas, Moller–Rabinovich

$\mathbf{MSO}^{\wp}/{\sim} \equiv \mathbf{CTL}^*$ over the class of all (unranked) trees

Janin–Walukiewicz

$\mathbf{MSO}/{\sim} \equiv \mathbf{L}_{\mu}$ over the class of all Kripke structures

common thread: **upgradings** between game-based equivalences

sensitivity to underlying class **e.g. FO/$\sim$**

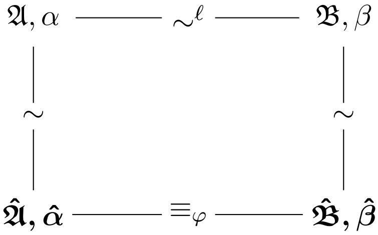
$\mathrm{FO}/{\sim} \equiv \mathrm{ML}$ over $\mathcal{C}$ $\not\Rightarrow$ $\mathrm{FO}/{\sim} \equiv \mathrm{ML}$ over $\mathcal{C}_0$ for $\mathcal{C}_0 \subseteq \mathcal{C}$

unless $\sim$ invariance over $\mathcal{C}_0$ does imply $\sim$ invariance over $\mathcal{C}$

crux: expressive completeness **e.g. FO/$\sim$**

φ invariant under $\sim$ on $\mathcal{C}$
$\Rightarrow$ φ invariant under $\sim^\ell$ on $\mathcal{C}$ for some ℓ
$\Rightarrow$ φ expressible in ML_ℓ over $\mathcal{C}$

upgrading idea:



examples from D/O LICS 05:

- **locality based**
  upgrading $\sim^\ell$ to some level of Gaifman equivalence
  $\mathbf{FO}/\!\sim\, \equiv \mathbf{ML}[\forall]$ on (finite) rooted frames

- **decomposition based**
  upgrading $\sim^\ell$ to $\equiv_q$ through path decomposition & pumping
  $\mathbf{FO}/\!\sim\, \equiv \mathbf{ML}$ on (finite) transitive $\prec$-trees

new decomposition & interpretation arguments:

- **transitive frames**, allowing reflexivity
- **finiteness vs. well-foundedness**
- **results for $\mathbf{MSO}/\!\sim$**

the point(s) of this talk

- **FO path decomposition & pumping argument**
  on irreflexive transitive trees: $\prec$-trees

- **extension via interpretation & upgrading**
  to reflexive transitive trees: $\preccurlyeq$-trees
  and other transitive frames, finite and infinite

- **extension to cover MSO**
  over transitive frames with well-foundedness constraints,
  collapse of MSO/$\sim$ to FO/$\sim$ and ramifications of

  de Jongh–Sambin–Smorynski
  Janin–Walukiewicz

finiteness vs. well-foundedness conditions

distinguish in transitive frames:

no infinite paths (⇒ no reflexive nodes)

⇒ no infinite irreflexive paths (⇒ no cycles) $R \setminus R^{\circ}$

⇒ no infinite irreversible paths $R \setminus R^{-1}$

finiteness ⇒ no infinite irreversible paths

path-finite transitive frames

no infinite strict/irreversible paths

no infinite nested chain of generated subframes

a quasi-wellordering property

$\prec$-trees: FO path decomposition & pumping argument

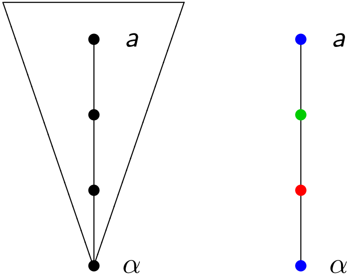
pass to wide companions $\quad \mathbf{s_q(\mathfrak{A}, \alpha) := \mathrm{TC}\big((\mathfrak{A} \otimes q)^*_\alpha\big)}$

boosted multiplicities

finitary saturation

tree-unfolding and transitive closure

colour with $\equiv_{q-1}$-classes of subtrees



for pumping argument along paths from root to node a

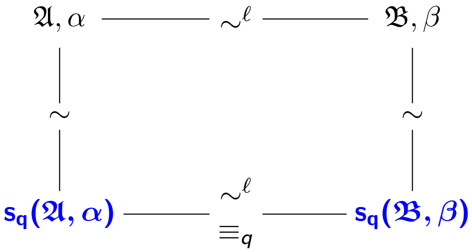
pumping lemma/Ehrenfeucht–Fraïssé

bound on length of relevant words realised

+ sub-word closure property in $s_q(\mathfrak{A}, \alpha)$ **(!)**

$\longrightarrow$ (non-elementary) bound on ℓ for $\sim^\ell$ that governs $\equiv_q$

the upgrading: $\varphi \in \mathbf{FO_q}/\!\sim \;\Rightarrow\; \varphi$ **invariant under** $\sim^\ell$



the (harmless) extension to $\preccurlyeq$-trees

via the natural quantifier-free interpretation: $\boldsymbol{\mathfrak{A}_{\preccurlyeq}, \alpha \mapsto \mathfrak{A}_{\prec}, \alpha}$

the upgrading:

$$
\begin{array}{ccccc}
\mathfrak{A}_{\preccurlyeq}, \alpha & \text{———} & \sim^{2\ell+1} & \text{———} & \mathfrak{B}_{\preccurlyeq}, \beta \\
| & & & & | \\
\sim & & & & \sim \\
| & & & & | \\
\tilde{\mathfrak{A}}_{\preccurlyeq}, \alpha & \text{———} & \sim^{2\ell+1} & \text{———} & \tilde{\mathfrak{B}}_{\preccurlyeq}, \beta \\
| & & & & | \\
\boldsymbol{\tilde{\mathfrak{A}}_{\prec}, \alpha} & \text{———} & \sim^{\ell} & \text{———} & \boldsymbol{\tilde{\mathfrak{B}}_{\prec}, \beta}
\end{array}
$$

stretching:
insertion of copies
of reflexive nodes

$\boldsymbol{\Rightarrow \tilde{\mathfrak{A}}_{\preccurlyeq}, \alpha \ \equiv_{\mathbf{q}} \ \tilde{\mathfrak{B}}_{\preccurlyeq}, \beta}$

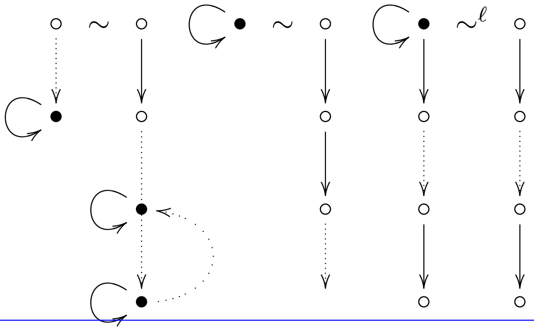
after finitary saturation

if (ir)reflexivity is not prescribed:

mistaken generalisation in D/O 05

$\varphi(x) = \exists y(Exy \wedge Eyy)$

- $\sim$ invariant over finite/path-finite transitive frames
- not $\sim$ invariant over transitive frames with infinite paths
- not $\sim^\ell$ invariant for any ℓ over all finite transitive frames



$\Rightarrow$ **FO/$\sim$ $\not\equiv$ ML over the class of all finite transitive frames**

extension to transitive tree-like frames $\quad$ FO/$\sim \;\equiv$ ML[$\Diamond^*$]

$\Diamond^*\varphi \equiv \exists y\big(Exy \wedge Eyy \wedge \varphi(y)\big)$ $\qquad$ with associated $\sim_*$ / $\sim_*^\ell$

via the natural quantifier-free interpretation: $\mathfrak{A},\alpha \;\mapsto\; \mathfrak{A}^\bullet_\prec,\alpha$
with marker predicate for reflexive nodes

the upgrading:

$$
\begin{array}{ccccc}
\mathfrak{A},\alpha & \text{———} & \sim_*^L & \text{———} & \mathfrak{B},\beta \\
| & & & & | \\
\sim & & & & \sim \\
| & & & & | \\
\tilde{\mathfrak{A}},\alpha & \text{———} & \sim_*^L & \text{———} & \tilde{\mathfrak{B}},\beta \\
| & & & & | \\
\tilde{\mathfrak{A}}^\bullet_\prec,\alpha & \text{———} & \begin{array}{c}\sim^\ell \\ \equiv_q\end{array} & \text{———} & \tilde{\mathfrak{B}}^\bullet_\prec,\beta
\end{array}
$$

$L = \ell^2 + \ell + 1$
non-trivial game analysis

extension to MSO/$\sim$ (base case: $\prec$-trees)

subtree decomposition rather than path decomposition

upgrading, for path-finite transitive $\prec$-trees $\mathfrak{A}, \alpha$, $\mathfrak{B}, \beta$:

$$\mathfrak{A}, \alpha \sim^{\mathsf{L}} \mathfrak{B}, \beta \quad \longrightarrow \quad \mathsf{s}_{\mathsf{Q}}(\mathfrak{A}), \alpha \equiv^{\mathsf{MSO}}_{\mathsf{q}} \mathsf{s}_{\mathsf{Q}}(\mathfrak{B}), \beta$$

boosted multiplicities
tree unfolding and transitive closure

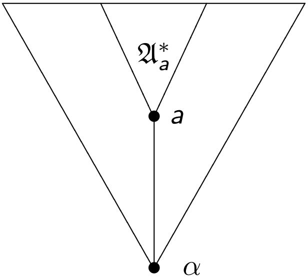
for suitable $\mathbf{L} = \mathbf{L}(\mathbf{q}), \mathbf{Q} = \mathbf{Q}(\mathbf{q})$

proof idea: in $\mathfrak{A}^* := \mathrm{s}_{\mathbf{Q}}(\mathfrak{A}) = \mathrm{TC}\big(\mathfrak{A} \otimes \mathbf{Q}\big)^*_{\alpha}$:

$\mathrm{tp}^{\mathsf{MSO}}_{q}(\mathfrak{A}^*_a)$ determined by $\mathbf{atp}(\mathbf{a})$ and ...

in general:

multiplicities of $\mathrm{tp}^{\mathsf{MSO}}_{q}(\mathfrak{A}^*_b)$
at direct $\prec$-successors b of a



here (due to saturation/transitivity)**:**

the **set** $\{\mathrm{tp}^{\mathsf{MSO}}_{q}(\mathfrak{A}^*_b) : a \prec b\}$

monotonicity $\Rightarrow$ finiteness

in $\mathfrak{A}^* := \mathsf{s}_{\mathbf{Q}}(\mathfrak{A}) = \mathrm{TC}\big(\mathfrak{A} \otimes \mathbf{Q}\big)^*_{\alpha}$

by induction on (finite) sets s of MSO_q-types

find $\xi_s(x) \in \mathrm{ML}_{|s|+1}$ s.t. **in path-finite trees**:

$$\boxed{\xi_s(x) = \text{“}\left\{\mathbf{tp}_q^{\mathsf{MSO}}(\mathfrak{A}^*_b) \colon x \prec b\right\} = s\text{”}}$$

well-foundedness

the upgrading:

$$\begin{array}{ccc}
\mathfrak{A}, \alpha & \text{———} \sim^{L(q)} \text{———} & \mathfrak{B}, \beta \\
| & & | \\
\sim & & \sim \\
| & & | \\
\mathsf{s}_{\mathbf{Q(q)}}(\mathfrak{A}) & \text{———} \equiv_q^{\mathsf{MSO}} \text{———} & \mathsf{s}_{\mathbf{Q(q)}}(\mathfrak{B})
\end{array}$$

$L(q) = \#\ q\text{-types} + 1$

results for MSO/$\sim$ over transitive frames

on (path-)finite $\prec$-trees (Löb frames) and
(path-)finite $\preccurlyeq$-trees (Grzegorczyk frames):

$$\mathbf{MSO}/\sim \;\equiv\; \mathbf{FO}/\sim \;\equiv\; \mathbf{ML}$$

on (path-)finite transitive frames:

$$\mathbf{MSO}/\sim \equiv \mathbf{FO}/\sim \equiv \mathbf{ML}[\Diamond^*]$$

translation transitive $\longrightarrow$ transitive tree-like $\longrightarrow$ $\prec$-trees:
via natural FO-interpretations as before

collapse results
de Jongh–Sambin–Smorynski / Janin–Walukiewicz

ramifications and new proofs of

de Jongh–Sambin–Smorynski:

$\mathbf{L}_\mu \equiv \mathbf{ML}[\Diamond^*]$
on (path-)finite transitive frames

generalisation from Löb frames/new proof

Janin-Walukiewicz:

$\mathbf{MSO}/\sim \; \equiv \; \mathbf{ML}[\Diamond^*] \subseteq \mathbf{L}_\mu{}^{\mathbf{1}} \subseteq \mathbf{L}_\mu$
on (path-)finite transitive frames

special case of an FMT variant/new proof

cf. ten Cate–Fontaine–Litak: finite Löb frames

thanks to: Balder ten Cate & Johan van Benthem

$\longrightarrow$ see preliminary full paper D/O 2008