Up To Bisimulation – but keep it finite!

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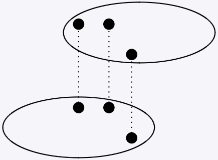
Global Finite Realisations of Local Specifications

examples of local views & specifications

- **exploded view of a hypergraph**



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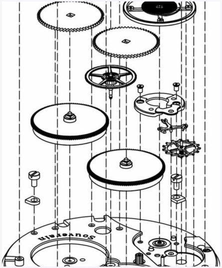
  . . . or a watch movement



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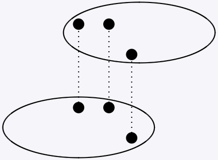
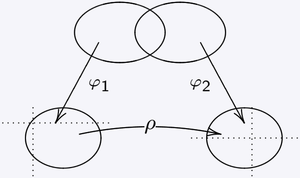


- **coordinate charts for manifolds**



- **decomposition and synthesis** of graphs, hypergraphs, ...

- **implicit specifications of (macro-)bisimulation types as in guarded extension properties** $\forall \mathbf{x}\big(\theta(\mathbf{x}) \rightarrow \exists \mathbf{y}\theta'(\mathbf{xy})\big)$

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

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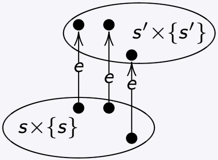
intersection graph of $\mathcal{A}$:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') \colon s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges $\mathbf{s} \in \mathbf{S}$

exploded view of $\mathcal{A}$ based on $\mathrm{I}(\mathcal{A})$

the disjoint union of the hyperedges $s \in S$
with partial bijections ρ_e for $e = (s, s') \in E$



$\rightsquigarrow$ **format of local overlap specifications**

overview

(I) specification & realisation of overlap patterns

(II) reduced products with groupoids (core results)

(III) from local to global symmetries

(IV) applications in modal & guarded logics

(I) abstract specification & realisation

incidence pattern $\mathrm{I} = (S, (E[s,s'])_{s,s' \in S})$

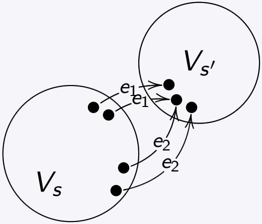
multi-graph with vertices $s \in S$ (sorts)
directed edges $e \in E[s,s']$ from s to s' with $e^{-1} \in E[s',s]$

- **fixed bisimulation type for pairwise overlaps**

I-**graph** $\mathrm{H} = \big(V, (V_s)_{s \in S}, (\rho_e)_{e \in E}\big)$

vertex set V partitioned into sorts V_s for $s \in S$
ρ_e a partial* bijection between V_s and $V_{s'}$ for $e \in E[s,s']$

- **an exploded view of the desired pairwise overlaps**



realisation:

a realisation of $\mathrm{H} = (V, (V_s), (\rho_e))$ is a
hypergraph $\mathcal{A} = (A, \tilde{S})$ with projection $\pi \colon \tilde{S} \longrightarrow S$
and an atlas of bijections $\pi_{\tilde{s}} \colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

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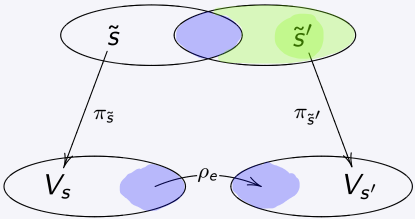
- **all specified overlaps are realised:**
  for $e \in E[s, s']$, ρ_e is realised at every $\tilde{s} \in \pi^{-1}(s)$ by an actual overlap with some $\tilde{s}' \in \pi^{-1}(s')$



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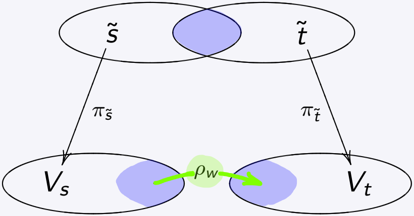
- **no further, incidental overlaps occur:**
  every actual overlap of $\mathrm{I}(\mathcal{A})$ is induced by a single
  composition ρ_w of partial bijections ρ_e in H

$w = e_1 \cdots e_n \in E^*[s, t]$
$\rho_w = \rho_{e_n} \circ \cdots \circ \rho_{e_1}$



realisations vs. exploded views

any hypergraph $\mathcal{A} = (A, S)$ is a realisation of its exploded view

- **$\mathcal{A}$ is reconstructed from its exploded view $\mathrm{H}(\mathcal{A})$ as a quotient $\mathrm{H}(\mathcal{A})/\!\approx$ w.r.t. $\approx$ induced by the identifications encoded in $\mathrm{H}(\mathcal{A})$**

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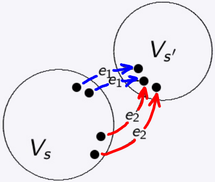
- in general $\mathrm{H}/\!\approx$ may fail to realise H:
  $\approx$ **may even collapse individual** $\mathbf{V_s}$



idea: try local unfolding in products of H **with ...?**

the role of groupoids/inverse semigroups

composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

groups capture global symmetries

groupoids capture local/partial symmetries

(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

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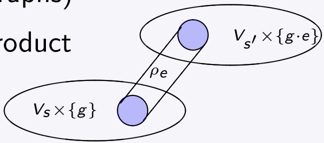
reduced products as candidate realisations:

$\rightsquigarrow$ $\mathbb{H} \times \mathbb{G}$ natural direct product (of I-graphs)

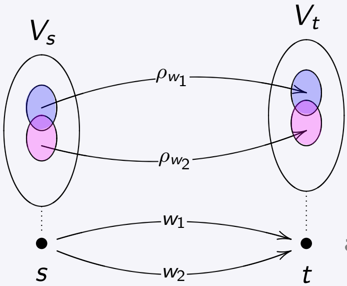
$\rightsquigarrow$ $\mathbb{H} \otimes \mathbb{G} := (\mathbb{H} \times \mathbb{G})/\approx$ reduced product



when is this a realisation of $\mathbb{H}$?

obstructions to simple realisations

- $\mathbb{H}$ may fail to be *coherent*:
  conflicting identifications collapsing individual V_s

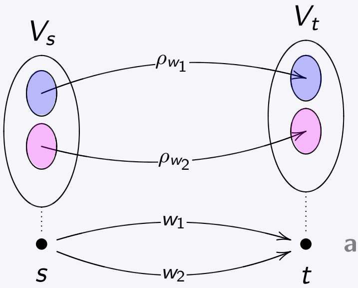


a bad cycle in $\mathbb{G}$?

can be overcome by relatively simple pre-processing

obstructions to simple realisations

- $\mathbb{H}$ and $\mathbb{G}$ may fail to be *confluent* in the product: incidental overlaps (with potential conflicts at the relational level)

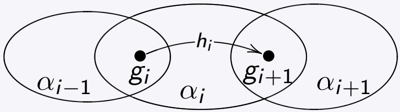


a bad cycle in $\mathbb{G}$?

$\rightsquigarrow$ **need substantial acyclicity conditions on $\mathbb{G}$**

an appropriate notion of acyclicity

- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



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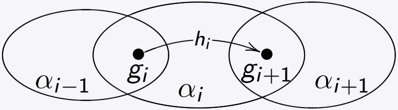
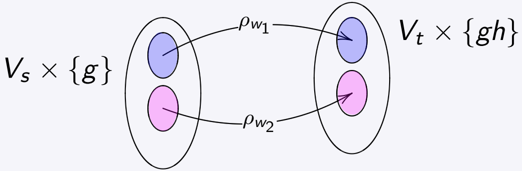
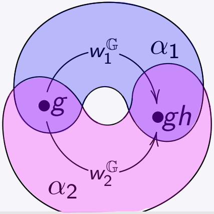


- in particular, need to avoid certain coset cycles of length 2



$$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$$



any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

idea: in an inductive construction generate $\mathbb{G}$ from (semi)group action on amalgamation chains that unfold short cosets cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H}$ (an I-graph), obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

symmetric realisations

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symmetric realisations

corollary

every finite hypergraph admits, for $\mathsf{N} \in \mathbb{N}$, finite coverings that

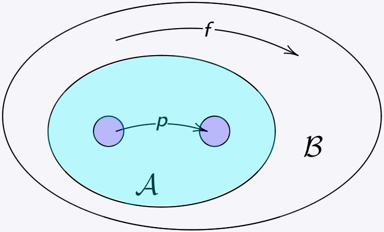
- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

(III) from local to global symmetries

extension property for partial automorphisms (EPPA):
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$



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theorem (Herwig–Lascar 00)

same, as a *finite model property* over any class $\mathcal{C}$
defined by finitely many forbidden homomorphisms

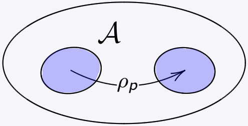
if $\mathcal{A} \in \mathcal{C}_{\mathrm{fin}}$ has any EPPA extension in $\mathcal{C}$
then it also has a finite one in $\mathcal{C}_{\mathrm{fin}}$

new proof of full Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification
for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I(\mathcal{A}, P)}$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H(\mathcal{A}, P)}$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!



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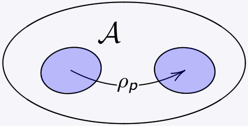

(iii) **symmetric realisations of $\mathbf{H(\mathcal{A}, P)}$ are EPPA extensions !**

(iv) **N-acyclic EPPA extensions are N-free:**
admit N-local homomorphisms into every (finite or infinite) EPPA extension due to their N-local tree-decomposability

(IV) applications in modal/guarded logics

- **characterisation theorems** (fmt)
  for guarded logics and relatives

  using finite coverings of controlled acyclicity

- **finite model properties & finite controllability for guarded logics and constraints**

  using finite coverings of controlled acyclicity
  and/or Herwig–Lascar extension properties

$\longrightarrow$ **O**_(LICS10&JACM13)
O_(APAL13)
Bárány–Gottlob–O_(LICS10&LMCS14)
Bárány–ten Cate–O_(VLDB12)
O_(LICS13&arXiv14/15)

characterisation theorems (fmt & classical)

theorem (O_10)

$\mathrm{GF} \equiv \mathrm{FO}/{\sim_{\mathrm{g}}}$ and $\mathrm{GF} \equiv_{\mathrm{fin}} \mathrm{FO}/{\sim_{\mathrm{g}}}$

characterisation theorems (fmt & classical)

theorem (O_10)

$\mathsf{GF} \equiv \mathsf{FO}/{\sim_{\mathrm{g}}}$ and $\mathsf{GF} \equiv_{\mathsf{fin}} \mathsf{FO}/{\sim_{\mathrm{g}}}$

idea: show that $\sim_{\mathrm{g}}$-invariance of $\varphi \in \mathrm{FO}^m$ implies $\sim_{\mathrm{g}}^{\ell}$-invariance for some $\ell = \ell(m)$ such that over suitable locally sufficiently acyclic (finite) structures, $\sim_{\mathrm{g}}^{\ell}$ refines $\equiv_{\mathsf{FO}}^{m}$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, dash, "\sim_{\mathrm{g}}^{\ell}" description] \arrow[d, dash, "\sim_{\mathrm{g}}" description] & & \mathcal{B} \arrow[d, dash, "\sim_{\mathrm{g}}" description] \\
\mathcal{A}^* \arrow[rr, dash, "\equiv_{\mathsf{FO}}^{m}" description] & & \mathcal{B}^*
\end{tikzcd}$$

adaptations: guarded negation fragment (O_13)
common knowledge extensions of ML (??)

finite model properties & finite controllability

theorem (Bárány–Gottlob–O_10/Rosati06)

finite controllability for union of conjunctive queries Q
w.r.t. constraint $\alpha \in \mathsf{GF}$:

$$\alpha \models Q \quad \Leftrightarrow \quad \alpha \models_{\mathsf{fin}} Q \qquad (\dagger)$$

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idea 1: via "treeification" Q^* of Q s.t. $Q \equiv_{\mathsf{acyc}} Q^* \in \mathsf{GF}$ and locally sufficiently acyclic (finite) unfoldings to show

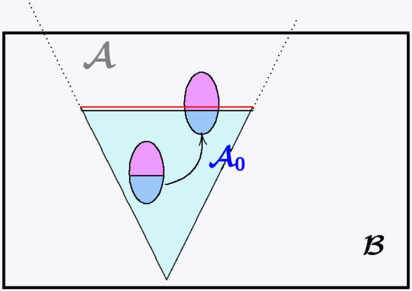
$$\alpha \models_{\mathsf{fin}} Q \;\Rightarrow\; \alpha \models_{\mathsf{fin}} Q^* \;\Leftrightarrow\; \alpha \models Q^* \;\Rightarrow\; \alpha \models Q.$$

idea 2: view $(\dagger)$ as a fmp for $\alpha \in \mathsf{GF}$ within $\mathcal{C} = \mathrm{Mod}(\neg Q)$, which follows from Herwig–Lascar EPPA (!)

guarded fmp (in $\mathcal{C} = \mathrm{Mod}(\neg\mathrm{Q})$) & EPPA

following an idea of Erich Grädel for GF, see (Grädel–O_14)

after relational Skolemisation use EPPA to obtain finite model as finite closure of finite substructure of infinite model w.r.t. guarded $\forall\exists$-requirements



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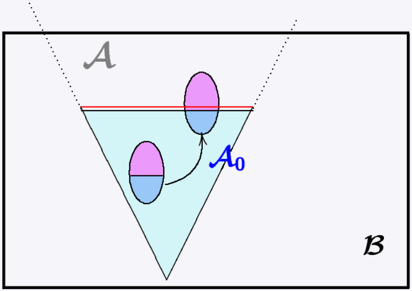

Herwig–Lascar EPPA yields finite controllability for UCQ
fmp within $\mathcal{C} = \mathrm{Mod}(\neg \mathrm{Q})$ as a qualitative result

summary

- a generic construction of
  highly acyclic finite groupoids

- a universal & generic route to the synthesis of
  finite realisations (and coverings) in reduced products

- symmetry and acyclicity of realisations supports
  extensions of local to global symmetry

- further applications in finite model theory

$\longrightarrow$ **Finite Groupoids, Finite Coverings & Symmetries in Finite Structures** (arXiv 2015 (v4))