Cayley structures ———— — generic Epistemic Models

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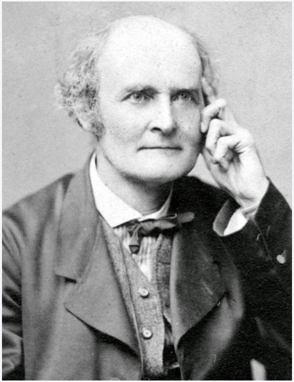


Arthur Cayley, 1821–1895

overview

- **epistemic modal logic & common knowledge**
- **basic algebra and combinatorics**

  Cayley structures & measures of acyclicity
- **uses in (finite) model theory**

  algebraic interpretations and back&forth
  in dual hypergraphs of Cayley structures

up to bisimulation: modal model theory

- bisimulation: modal back&forth, modal Ehrenfeucht–Fraïssé

- expressive completeness/characterisations:

  van Benthem–Rosen: $\mathrm{ML} \equiv \mathrm{FO}/{\sim}$

  Janin–Walukiewicz: $\mathrm{L}_\mu \equiv \mathrm{MSO}/{\sim}$

- fmt characterisations

we do have	vs.	we don't have
$\mathrm{ML} \equiv \mathrm{FO}/{\sim}$	vs.	$\mathrm{L}_\mu \equiv \mathrm{MSO}/{\sim}$
many variants		just classically?

Kripke structures

- **Kripke structures:** edge- and vertex-coloured graphs

  vertices: **possible worlds**
  colours for *basic propositions* $p \in \Phi$

  edges: **accessibility relations**
  colours for *agents* $a \in \Gamma$

- **the epistemic setting: equivalence frames (S5)**

  a-equivalence classes $[w]_a$: clusters of worlds w.r.t. *observational equivalence* for agent a

  modelling factual uncertainty

epistemic modal logic & common knowledge

- **basic modal logic ML:**

  modalities $\Box_a, \Diamond_a$ for $a \in \Gamma$ (along a-edges)

  $[\Box_a \varphi](x) \equiv \forall y \big(R_a xy \rightarrow [\varphi](y) \big)$

  agent a knows φ

- **common knowledge logic ML[CK]:**

  modalities $\Box_\alpha, \Diamond_\alpha$ for $\alpha \subseteq \Gamma$ (along α-paths)

  $[\Box_\alpha \varphi](x) \equiv \forall y \big([\bigcup_{a \in \alpha} R_a]^* xy \rightarrow [\varphi](y) \big)$

  φ *is common knowledge among all* $a \in \alpha$

**NB: reachability closure for $\Box_\alpha \varphi \equiv \bigwedge_{a_1 \dots a_n \in \alpha^*} \Box_{a_1} \cdots \Box_{a_n} \varphi$
is MSO-definable and corresponds to greatest fixpoint**

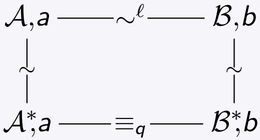
expressive completeness

- **ML $\equiv$ FO/$\sim$ (classically and in fmt)**

expressive completeness is a compactness property
$\left\{ \begin{array}{l} \text{viz.} \quad \varphi \in \text{FO } \sim\text{-invariant} \;\Rightarrow \\ \varphi \; \sim^{\ell}\text{-invariant for some finite } \ell \end{array} \right.$

through *upgrading* argument:

force strategy in first-order E–F game over suitable (finite) $\sim$-companions



Gaifman locality of FO

- **$\mathsf{L}_\mu \equiv$ MSO/$\sim$ (classically only?)**

expressive completeness by global collapse for automata-theoretic model-checking in suitable (infinite) $\sim$-companions, viz. trees

non-locality of MSO

common knowledge in CK structures

CK structure: expansion of S5 (equivalence) structure by the accessibility relations for common knowledge

$$R_\alpha = [\bigcup_{a \in \alpha} R_a]^* \quad \text{for } \alpha \subseteq \Gamma \quad \text{so that}$$

ML[CK] over **S5 structures**	$\equiv$	**ML** over **CK structures**

non-elementary

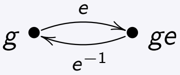
idea: use Cayley structures as special CK structures

control combinatorics algebraically

Cayley groups & graphs

group $G = (G, \cdot, 1)$ generated by $E = E^{-1}$ s.t. $G = \langle E \rangle^G$
$\rightsquigarrow$ graph representation:

$$\big(G, (R_e)_{e \in E}\big) \;\text{ where } R_e = \{(g, ge) \colon g \in G\}$$



examples:

- $\mathbb{Z}_n$ with generators $1, -1$
  $\rightsquigarrow$ bi-directed n-cycle
- group with k involutive generators $e_i = e_i^{-1}$
  $\rightsquigarrow$ k-regular labelled undirected tree

Cayley structures as special CK structures

Cayley structure: based on Cayley graph with generator set E partitioned among agents

$$E = \dot{\bigcup}_{a \in \Gamma} E_a$$

and induced equivalence relations

$$R_a = \big\{(g, gh) \colon h \in \langle E_a \rangle \subseteq G \big\}$$

$$R_\alpha = \big\{(g, gh) \colon h \in \langle \textstyle\bigcup_{a \in \alpha} E_a \rangle \subseteq G \big\} = [\textstyle\bigcup_{a \in \alpha} R_a]^*$$

$\boldsymbol{E_a}$ **: generator set for agent** $\boldsymbol{a}$**'s uncertainty**
$\boldsymbol{E_\alpha = \bigcup_{a \in \alpha} E_a}$ **: generator set for shared uncertainty in** $\boldsymbol{\alpha}$

$\boldsymbol{\alpha}$**-equivalence = membership in same** $\boldsymbol{\langle E_\alpha \rangle}$**-coset**

algebraic, highly regular ——
yet generic/$\sim$ (see below)

back to upgrading arguments

idea: in suitable Cayley structures upgrade $\sim^\ell$ to $\equiv_q$

$$\begin{array}{ccc} \mathcal{A}, a & \text{———} \sim^\ell \text{———} & \mathcal{B}, b \\ | & & | \\ \wr & & \wr \\ | & & | \\ \mathcal{A}^*, a & \text{———} \equiv_q \text{———} & \mathcal{B}^*, b \end{array}$$

issues:

- disagreement on short cycles
- disagreement on small multiplicities
- locality? — connected components are Gaifman cliques (!)
  need to match small/large distances at different scales (!)

acyclicity criteria in Cayley groups

to control FO_q through $\sim^\ell$, need to avoid short cycles

- large girth, **no short generator cycles**,
  the classical graph-theoretic notion

  → **finite Cayley graphs of girth $\geqslant N$**
  from simple permutation group action on finite trees

- much stronger: **no short coset cycles**,
  a hypergraph-inspired notion from O_12

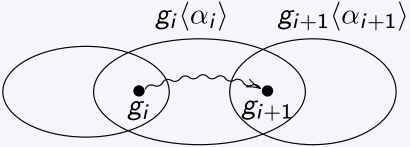
  → **finite N-acyclic Cayley groups**
  from intricate interleaving of finite unfoldings with permutation group actions

→ technical notes at the end

coset acyclicity

coset cycle of length n in $G = \langle E \rangle$: $\quad (g_i, g_i\langle\alpha_i\rangle)_{i \in \mathbb{Z}_n}$:

cycle of $\langle\alpha_i\rangle$-cosets
for subsets $\alpha_i \subseteq E$



N-acyclic Cayley groups (O_12)**:** for any $N \in \mathbb{N}$ and generator set find finite Cayley group without coset cycles of lengths $\leqslant N$

coset acyclicity is α-acyclicity of a dual hypergraph:

vertices : cosets $g\langle\alpha\rangle$ for $g \in G, \alpha \subseteq E$
hyperedges : clusters $[g] = \{g\langle\alpha\rangle : \alpha \subseteq E\}$

Cayley structures as generic CK structures

theorem (Canavoi–O_17)**:**

every (finite) S5/CK structure admits bisimilar coverings by (finite) N-acyclic Cayley structures, for any N

ML[CK] over **S5 structures**	$\equiv$	**ML** over **CK structures**
$\vert\vert\vert$	**up to** $\sim$	$\vert\vert\vert$
ML[CK] over **Cayley structures**	$\equiv$	**ML** over **Cayley structures**

as acyclic and as highly branching as desired

gain high algebraic regularity

back to upgrading arguments

task: in sufficiently acyclic, highly branching Cayley structures upgrade $\sim^\ell$ to $\equiv_q$

$$
\begin{array}{ccc}
\mathcal{A}, a & \overset{}{\text{——} \sim^\ell \text{——}} & \mathcal{B}, b \\
| \wr & & | \wr \\
\mathcal{A}^*, a & \text{——} \equiv_q \text{——} & \mathcal{B}^*, b
\end{array}
$$

issues:

- disagreement on short cycles ✓
- disagreement on small multiplicities ✓
- locality? — connected components are Gaifman cliques (!) need to match small/large distances at different scales (!)

upgrading & game over Cayley structures

in sufficiently acyclic and highly branching Cayley structures:

- in q-round FO Ehrenfeucht–Fraïssé game can maintain isomorphic local tree decompositions in dual hypergraphs
- algebraic regularity of N-acyclic Cayley structure allows to match small distances at all scales
- e.g. coset acyclicity at level N yields
  – *unique least connecting set of agents* (for $N \geqslant 2$)
  – *unique core paths witnessing distance d* (for $N \geqslant f(d)$)

non-trivial structure theory for N-acyclic Cayley structures and their dual hypergraphs
$\rightsquigarrow$ **Felix Canavoi's forthcoming dissertation**

dual viewpoints: how acyclicity comes in

- plain S5: boost girth in dual graph (Dawar–O_09)

$S5$ Kripke	$\leftrightsquigarrow$	vertex-coloured graph
a-classes $[w]_a$	$\leftrightsquigarrow$	vertices
intersections/worlds	$\leftrightsquigarrow$	edges
N-acyclic hypergraph of equivalence classes	$\leftrightsquigarrow$	girth $\geqslant N$

$\Downarrow$ **FO int.**

- CK/Cayley: boost coset acyclicity (Canavoi–O_17)

Cayley CK	$\leftrightsquigarrow$	dual hypergraph
α-classes $[w]_\alpha$	$\leftrightsquigarrow$	vertices
intersections/worlds	$\leftrightsquigarrow$	hyperedges
coset N-acyclicity	$\leftrightsquigarrow$	hypergraph N-acyclicity

alg. int. $\Downarrow$

results & directions

theorem (Canavoi–O_17)**:**

ML[CK] $\equiv$ ML $\equiv$ FO/$\sim$ in (finite) CK structures

ML[CK] $\equiv$ FO[CK]/$\sim$ in (finite) S5 structures

new results/work in progress:

analogous characterisations even at the level of
relativised common knowledge and *public announcement*

further potential:

explore richer "**algebraic interpretations**" in Cayley structures
to cover other reachability phenomena in "**almost FO**"

technical notes (1): the basic Cayley covering

from S5/CK structure to bisimilar covering by Cayley structure

with connected S5 structure $\mathcal{K} = (K, (R_a)_{a\in\Gamma})$ associate permutation group action generated by $E := \dot{\bigcup}_{a\in\Gamma} R_a$ via

$$e = (u, u') \quad \rightsquigarrow \quad \pi_e = (u, u') \in \mathrm{Sym}(K)$$

individual edges $\rightsquigarrow$ swaps of related worlds

- $G := \langle \pi_e \colon e \in E \rangle \subseteq \mathrm{Sym}(K)$

  to boost branching degree, put multiple copies of each e

- $G(\mathcal{K}) := \big(G, (\{(g, gh) \colon h \in \langle R_a \rangle\})_{a\in\Gamma}\big)$,

  $g \longmapsto u \cdot g$, for fixed source $u \in K$, induces bisimilar covering $G(\mathcal{K}) \longrightarrow \mathcal{K}$, compatible with passage to the R_α for $\alpha \subseteq \Gamma$ (!)

technical notes (2): N-acyclic Cayley groups

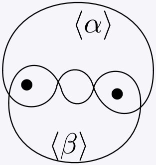
for given set E and $N \geqslant 2$ find $G = \langle E \rangle^G$
without coset cycles of length $\leqslant N$

by induction w.r.t. $k < |E|$ find G_k s.t.

- for $\alpha \subseteq E$ of size $\leqslant k$: $\langle \alpha \rangle^{G_{k+1}} \simeq \langle \alpha \rangle^{G_k}$
- for $\alpha \subseteq E$ of size $\leqslant k+1$: $\langle \alpha \rangle^{G_{k+1}}$ N-acyclic

idea: abstract G_{k+1} from permutation group action on depth N tree unfolding of G_k w.r.t. subgroups $\langle \alpha \rangle$ for $|\alpha| \leqslant k$

NB: 2-acyclicity, $\langle \alpha \rangle \cap \langle \beta \rangle = \langle \alpha \cap \beta \rangle$,
is a discrete form of simple connectivity: not



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