Logics for Bisimulation Invariance

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like

- **ML $\equiv$ FO/$\sim$ (van Benthem 83)**
- **L$_\mu$ $\equiv$ MSO/$\sim$ (Janin-Walukiewicz 96)**
- **ML $\equiv$ FO/$\sim$ (fmt) (Rosen 97)**
- **... some others, and a big ?**

bisimulation & bisimulation invariance

bisimulation: $\mathcal{A}, a \sim \mathcal{B}, b$

- b&f equivalence between states in

  Kripke structures / transition systems
  informational (epistemic) equiv. / behavioural equiv.

- infinitary in nature: a strategy in unbounded b&f game
  with finite approximations $\sim^{\ell}$: a strategy for ℓ-rounds

$\sim$-invariance:

- whenever $\mathcal{A}, a \sim \mathcal{B}, b$ then $\mathcal{A}, a$ has $\mathcal{P}$ iff $\mathcal{B}, b$ has $\mathcal{P}$

- $\sim$-closed (rather than just $\simeq$-closed) classes of structures

study $\sim$-invariant properties and their definability

$\sim$-invariant properties

$\sim$-invariance undecidable (where non-trivial), **e.g.:**

- $\mathsf{FO}/{\sim} = \{\varphi(x) \in \mathsf{FO} \colon \varphi \sim\text{-invariant}\}$
  $\subsetneq \{\varphi(x) \in \mathsf{FO} \colon \varphi \sim\text{-invariant in finite models}\} \subsetneq \mathsf{FO}$

  both undecidable as syntactic fragments of FO
  (recursively inseparable, like VAL and FINVAL)

- $\mathrm{Ptime}/{\sim} = \{\mathcal{P} \in \mathrm{Ptime} \colon \mathcal{P} \text{ "respects } \sim\text{"}\}$

  undecidable (as a class of polynomially clocked TM say)
  (just like $\mathrm{Ptime}/{\simeq}$!)

and yet there can be effective syntax, e.g.:

- **$\mathsf{FO}/{\sim} \equiv \mathsf{ML}$** (van Benthem 83, and (fmt) Rosen 97)
- $\mathrm{Ptime}/{\sim} \equiv \mathsf{L}^{\omega}_{\mu}$, **a logic for $\sim$-invariant Ptime** (O_99)

(0) e.g., van Benthem–Rosen

FO/$\sim$ $\equiv$ ML

means that t.f.a.e.: (i) $\varphi(x) \in$ FO is $\sim$-invariant

(ii) $\varphi \equiv \psi$ for some $\psi \in$ ML

(ii) $\Rightarrow$ (i) direct from modal Ehrenfeucht–Fraïssé:

$\mathcal{P}$ definable in ML_ℓ $\Leftrightarrow$

$\mathcal{P}$ closed under $\sim^\ell$ $\Rightarrow$ $\mathcal{P}$ closed under $\sim$

(i) $\Rightarrow$ (ii) **(expressive completeness)** is equivalent to

$\mathcal{P}$ closed under $\sim^\ell$ $\Leftarrow$ $\mathcal{P}$ closed under $\sim$

for FO-definable $\mathcal{P}$ and suitable finite $\ell(\mathcal{P})$

and has a nice elementary, constructive proof . . .

. . . a more uniform upgrading strategy

$$
\begin{array}{ccc}
\mathcal{A}, a & \text{———} \sim^{\ell} \text{———} & \mathcal{B}, b \\
| & & | \\
\sim & & \sim \\
| & & | \\
\mathcal{A}^*, a & \text{——} \simeq_q / \equiv_q \text{——} & \mathcal{B}^*, b
\end{array}
$$

for suitable $\ell = \ell(q)$

possible obstructions:

differences w.r.t. (i) **small multiplicties**

(ii) **short cycles**

which are not controlled by any level of $\sim$

- **need** to avoid both
- **can** avoid both in products with finite (!) auxiliary structures

some variations obtained this way (Dawar–O_09):

- FO/$\sim$ over (finite) symmetric frames ($\sim = \sim_{\pm}$)
- FO/$\sim$ over (finite) rooted frames ($\sim = \sim_{\forall}$)
- FO/$\sim$ over (finite) multi-agent S5 frames (epistemic frames)

. . . make essential use of suitable (finite) graph coverings and FO-Gaifman locality

so do other variations not discussed here, for richer notions of bisimulation, like guarded bisimulation & guarded negation bisimulation, also of some interest in database context (e.g., Gottlob–Bárány–O_14, Bárány–tenCate–O_12)

(1) e.g., (finite) multi-agent epistemic S5 frames

$\mathcal{K} = (W, (E_i), (P_q))$

with equivalence relation E_i for each agent $i \in I$, with classes $[w]_i$ (i-edges form disjoint union of cliques, one for each class $[w]_i$)

can be untangled in finite (!) bisimilar coverings

$$\begin{array}{rcl} \pi\colon \mathcal{K}^* & \longrightarrow & \mathcal{K} \\ w^* & \longmapsto & \pi(w^*) \end{array}$$

with $\mathrm{graph}(\pi)$ as a bisimulation, and s.t. in $\mathcal{K}^*$

- $|[w^*]_i \cap [u^*]_j| \leqslant 1$ for $i \neq j$
- all non-trivial cycles (of $[w^*]_i$) are long

over such sparse companions, $\sim^\ell$ gives sound advice for $\simeq_q$

(2) one level up, the inquisitive setting

(Roelofsen–Ciardelli 11, Ciardelli 16)

plain modal S5:

knowledge/information states (of agent i) are *sets of worlds* $[w]_i$

that are equivalent/indistinguishable as far as agent i can tell

inquisitive S5:

questions/issues (of agent i) are *sets of sets of worlds*
described by downward closed collections in $\mathcal{P}([w]_i)$

possible, alternative resolutions from point of view of agent i

while plain modal models work with worlds and relational encodings of set assignments $w \mapsto [w]_i$ to worlds, now need assignments of sets of sets of worlds

... where MSO enters locally (Ciardelli–O_ytbd):

upgrading $\sim^\ell$ to $\simeq_q$ in two-sorted relational encodings of (finite) inquisitive S5 models

sort 1: states/worlds as usual

sort 2: information states at w, sets of worlds from individual $[w]_i$

with $\in$ and inquisitive assignments as mixed-sorted relations

uses combination of

- global pre-processing w.r.t. $[w]_i$-structure as before
- local post-processing within individual $[w]_i$

to control FO-game + local MSO-game over sort 1 (which is the FO-game over mixed sorts)

$\rightsquigarrow$ **$\sim$-invariance $\Rightarrow$ $\sim^\ell$-invariance for ...**

(3) aside on results for transitive frames (Dawar–O_09):

where MSO enters globally, in a tame manner

$R = R^*$(or R^+) makes accessible *future lightcone* in one step
$\rightsquigarrow$ look at rooted transitive frames ($\sim = \sim_\forall$)

rooted, transitive, (wellfounded, Löb and Gregorczyk), . . .
form non-elementary frame classes, in classical setting

R-related pairs of worlds $\rightsquigarrow$ connecting paths/words

q-tuples of worlds $\rightsquigarrow$ connecting tree

FO-analysis $\rightsquigarrow$ MSO-decomposition techniques

allowing reflexivity adds non-trivial structure: local clusters/cliques
$\rightsquigarrow$ **in fmt (only!), FO/$\sim$ calls for a nonstandard modality $\Diamond^*/\Box^*$**

in crucial finite/wellfounded scenarios: $\mathbf{FO}/\sim \equiv \mathbf{ML}^* \equiv \mathbf{MSO}/\sim$

(4) common knowledge epistemic S5 (with Felix Canavoi)

common knowledge modality: a fixpoint construct for reachability along paths accessible to groups of agents

$\Box^*_\alpha$**: "i_1 knows that i_2 knows that i_3 knows . . . "**, for all $i_j \in \alpha$, captures the idea of common knowledge among agents in α,

its dual $\Diamond^*_\alpha$ is based on the *transitive closure* of $\bigcup_{i\in\alpha} E_i$

clearly not first-order,
not amenable to classical techniques,
even in the classical setting

but $(\bigcup_{i\in\alpha} E_i)^*$ is all that's missing, to put $\Box^*_\alpha, \Diamond^*_\alpha$ in FO:

look at FO*: FO with access to $(\bigcup_{i\in\alpha} E_i)^*$, or

FO over models that interpret new relations E^*_α as $(\bigcup_{i\in\alpha} E_i)^*$

a non-elementary class of models

(4) common knowledge epistemic S5 (with Felix Canavoi)

FO*: FO over CK-models with relations $E^*_\alpha = (\bigcup_{i \in \alpha} E_i)^*$

theorem: **ML[CK]** $\equiv$ **FO*/$\sim$** (classically and fmt)

uses upgrading of $\sim^\ell$ to $\simeq_q$ in
the class of (finite) CK-models

based on (finite coverings by) Cayley groups

- generated by involutions $(g_{i,m})_{i \in I, 1 \leqslant m \leqslant M}$
- without short cycles formed by cosets
  of subgroups of the form $\langle g_{i,m} \colon i \in \alpha \rangle$

these cosets are the E^*_α-classes (!), and M is for multiplicity:
can control FO-game by (dual) game on coset hypergraph

$\rightsquigarrow$ **$\sim$-invariance $\Rightarrow$ $\sim^\ell$-invariance for ...**

other results on $\sim$-invariance between FO and MSO

mostly quite different techniques, and not fmt!

- Hafer–Thomas 87: monadic path logic / CTL^* on binary trees
- Moller–Rabinovich 03: monadic path logic / CTL^* on trees
- Carreiro 15: weak generalised monadic path logic / PDL
- (with Colin Hirsch 00): MSO on regular trees / L_μ

. . . and some even beyond MSO (but also not fmt)

- Grädel–Hirsch–O_02: $\mathrm{GSO}/{\sim_{\mathrm{g}}} \equiv \mu\mathrm{G}$ ("guarded J–W")