Bisimulation and Games

– for Modal and Guarded Logics

– for Graphs and Hypergraphs

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issues – in logic and combinatorics

- **what is modal/graph bisimulation good for?**
- **how does it generalise from graphs to hypergraphs?**
- **what is guarded/hypergraph bisimulation good for?**
- **which features and applications generalise?**

$\longrightarrow$ **logic vs combinatorial challenges**

organisation in four parts

(I) fairly classical introduction:
bisimulation and back&forth games
bisimulation as modal Ehrenfeucht–Fraïssé

(II) fairly classical applications:
bisimulation and the finite model theory of modal logics

(III) combinatorics of finite coverings:
bisimilar coverings for graphs and hypergraphs

(IV) more recent applications:
bisimulation and the finite model theory of guarded logics

I: bisimulation – the quintessential back&forth

on graph-like structures

Kripke structures (possible worlds/accessibility),
transition systems (states/transitions),
game graphs (positions/moves)

capture behavioural equivalence

in the sense of indistinguishability of worlds, states, positions, ...
w.r.t. alternating sequences of accessibility, transitions, moves, ...

core idea: dynamic b&f probing of possibilities

$\longrightarrow$ **dynamic exploration of structures that are static images of dynamic behaviour**

bisimulation game & bisimulation relations

the game:

2-person game $\begin{cases} \text{player } \mathbf{I}\text{: challenge} \\ \text{player } \mathbf{II}\text{: response} \end{cases}$

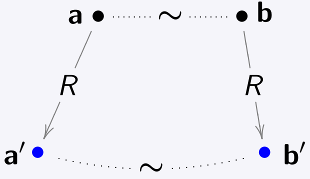
play over transition systems $\begin{cases} \mathcal{A} = (A, \mathbf{R}^{\mathcal{A}}, \mathbf{P}^{\mathcal{A}}) \\ \mathcal{B} = (B, \mathbf{R}^{\mathcal{B}}, \mathbf{P}^{\mathcal{B}}) \end{cases}$

positions: pairs (a, b) – correspondence between pebbled vertices

single round: challenge/response

I moves pebble in $\mathcal{A}$ or $\mathcal{B}$ along R-edge

II must do likewise in opposite structure



II loses in position (a, b) unless $a \sim^0 b$ (same colours)

I/**II** lose when stuck

bisimulation game & bisimulation relations

winning regions define **bisimulation equivalences:**

$\mathcal{A}, a \sim^{\ell} \mathcal{B}, b$	**II** has a winning strategy for ℓ rounds from (a, b)
$\mathcal{A}, a \sim^{\omega} \mathcal{B}, b$	**II** has a winning strategy for any finite no. of rounds from (a, b)
$\mathcal{A}, a \sim^{\infty} \mathcal{B}, b$	**II** has a winning strategy for infinite game from (a, b)

winning strategies in relational formalisation:

$Z \subseteq A \times B$ or
$(Z_m \subseteq A \times B)_{m \in \mathbb{N}}$
$(Z_m \subseteq A \times B)_{m \leqslant \ell}$

bisimulation relations with characteristic b&f requirements

bisimulation game & bisimulation relations

a bisimulation relation $Z \subseteq A \times B$

with characteristic b&f requirements

(*back*) for $(a, b) \in Z$ and $(b, b') \in R^{\mathcal{B}}$ there is $a' \in A$ s.t. $(a, a') \in R^{\mathcal{A}}$ and $(a', b') \in Z$

(*forth*) for $(a, b) \in Z$ and $(a, a') \in R^{\mathcal{A}}$ there is $b' \in B$ s.t. $(b, b') \in R^{\mathcal{B}}$ and $(a', b') \in Z$

witnesses existence of winning strategy from (a, b) in infinite game for any $(a, b) \in Z$

b&f systems $(Z_m)_{m \leqslant \ell}$ or $(Z_m)_{m \in \mathbb{N}}$

with stratified b&f conditions
analogously encode winning advice
for m rounds from $(a, b) \in Z_m$

classical motif: Ehrenfeucht–Fraïssé

pebble games for FO and FO$_\infty$

I and **II** over relational structures $\mathcal{A} = (A, \mathbf{R}^\mathcal{A})$ and $\mathcal{B} = (B, \mathbf{R}^\mathcal{B})$

positions: local isomorphisms $p \colon \mathbf{a} \mapsto \mathbf{b}$, $p \colon \mathcal{A}{\restriction}\mathbf{a} \simeq \mathcal{B}{\restriction}\mathbf{b}$

single round: challenge/response for
extension by one new pebble pair
$(p \colon \mathbf{a} \mapsto \mathbf{b}) \;\; \rightsquigarrow \;\; (p' \colon \mathbf{a}a' \mapsto \mathbf{b}b')$

winning regions:
b&f equivalences

$$\begin{cases} \mathcal{A}, \mathbf{a} \simeq^\ell \mathcal{B}, \mathbf{b} & \ell \text{ rounds} \\ \mathcal{A}, \mathbf{a} \simeq^\omega \mathcal{B}, \mathbf{b} & \text{finitely many rounds} \\ \mathcal{A}, \mathbf{a} \simeq^\infty \mathcal{B}, \mathbf{b} & \text{infinite game} \end{cases}$$

linked to levels of indistinguishability in first-order logic FO and its infinitary variant FO$_\infty$

$\simeq^\infty$ **classically known as** $\simeq_{\mathrm{part}}$**/partial isomorphy**

Ehrenfeucht–Fraïssé

Ehrenfeucht–Fraïssé/Karp thms

$\mathcal{A}, \mathbf{a} \simeq^{\ell} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv^{\ell}_{\mathsf{FO}} \mathcal{B}, \mathbf{b}$ *	qfr-depth ℓ FO-equiv.
$\mathcal{A}, \mathbf{a} \simeq^{\omega} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv_{\mathsf{FO}} \mathcal{B}, \mathbf{b}$ *	full FO equiv.
$\mathcal{A}, \mathbf{a} \simeq^{\infty} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv^{\infty}_{\mathsf{FO}} \mathcal{B}, \mathbf{b}$	FO_∞ equiv.

observations/proof ingredients:

- the sets $Z_m := \big\{(p\colon \mathbf{a} \mapsto \mathbf{b})\colon \mathcal{A}, \mathbf{a} \equiv^m_{\mathsf{FO}} \mathcal{B}, \mathbf{b}\big\}$
  satisfy b&f conditions
- **I** can force $\mathcal{A}, \mathbf{a} \not\equiv^m_{\mathsf{FO}} \mathcal{B}, \mathbf{b} \;\rightsquigarrow\; \mathcal{A}, \mathbf{a}a' \not\equiv^{m-1}_{\mathsf{FO}} \mathcal{B}, \mathbf{b}b'$
- existence of strategy for m rounds in game versus $\mathcal{A}, \mathbf{a}$
  is FO definable at qfr depth m (nested b&f conditions)*

  $*$ for finite relational vocabulary s.t. $\simeq^m$ has finite index

bisimulation & basic modal logic ML

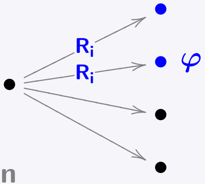
on graph-like structures

with binary (transition) relations $\mathbf{R} = (R_1, \ldots)$ $\rightsquigarrow$ modalities $\Diamond_i / \Box_i$
and unary (state) predicates $\mathbf{P} = (P_1, \ldots)$ $\rightsquigarrow$ basic propositions p_i

atomic formulae: $\bot, \top$ and p_i
booleans connectives: $\wedge, \vee, \neg$
modal quantification:

$$\Diamond_i\, \varphi \;\equiv\; \exists y \big(R_i xy \wedge \varphi(y) \big)$$

$$\Box_i\, \varphi \;\equiv\; \forall y \big(R_i xy \rightarrow \varphi(y) \big)$$



relativised FO quantification

observation

- local bisimulation condition ($\sim^0$) matches atomic ML-equiv.
- bisimulation b&f matches modal quantification pattern

bisimulation — modal Ehrenfeucht–Fraïssé

modal Ehrenfeucht–Fraïssé/Karp thms

$\mathcal{A}, a \sim^{\ell} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv^{\ell}_{\mathsf{ML}} \mathcal{B}, b$ *	ML-equiv./nesting depth ℓ
$\mathcal{A}, a \sim^{\omega} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv_{\mathsf{ML}} \mathcal{B}, b$ *	full ML equiv.
$\mathcal{A}, a \sim^{\infty} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv^{\infty}_{\mathsf{ML}} \mathcal{B}, b$	ML_{∞} equiv.

classically/modally:
when does $\simeq^{\omega}$ $\left(= \bigcap_{\ell} \simeq^{\ell}\right)$ coincide with $\simeq^{\infty}$ / $\simeq_{\mathrm{part}}$?
when does $\sim^{\omega}$ $\left(= \bigcap_{\ell} \sim^{\ell}\right)$ coincide with $\sim^{\infty}$ / $\sim$?

(modal) Hennessy–Milner thm

for suitably saturated $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A}, a \sim^{\omega} \mathcal{B}, b \quad \Rightarrow \quad \mathcal{A}, a \sim^{\infty} \mathcal{B}, b$

- **finitely branching**
- **modal- or ω-saturated**
- **recursively saturated pairs**

variations

- **two-way and global bisimulation $\approx$**

  add corresponding move options &
  extend challenge/response protocol

- **bisimulation in game graphs for other logics**

  states: admissible assignments
  transitions: quantification patterns

  all Ehrenfeucht–Fraïssé games are bisimulation games

- **hypergraph/guarded bisimulation $\rightarrow$ parts III/IV**

II: model theory of modal logics

in this section:

- tree model property
- finite model property
- descriptive complexity **(fmt)**
- expressive completeness **(classical and fmt)**

modal model theory = bisimulation invariant model theory

tree unfoldings

tree unfolding $\mathcal{A}_a = (A, \mathbf{R}^{\mathcal{A}}, \mathbf{P}^{\mathcal{A}}, a) \rightsquigarrow \mathcal{A}_a^* = (A_a^*, \mathbf{R}_a^*, \mathbf{P}_a^*, a)$

A_a^*: the set of all labelled directed paths w from a in $\mathcal{A}$ with projection $\pi \colon w \longmapsto \pi(w) \in A$, the endpoint of w

$R_a^* = \{(w, wRa') \colon (\pi(w), a') \in R^{\mathcal{A}}\}$

$P_a^* = \pi^{-1}(P^{\mathcal{A}})$

$\pi \colon \mathcal{A}_a^* \longrightarrow \mathcal{A}$ is an example of a bisimilar covering:

- **π is a homomorphism:** the forth-property for $\mathrm{graph}(\pi)$
- **π has lifting property:** the back-property for $\mathrm{graph}(\pi)$

a homomorphism inducing a bisimulation

$$\mathrm{graph}(\pi) = \{(w, \pi(w)) \colon w \in A_a^*\}$$

tree unfoldings and tree model property

$$\left.\begin{array}{r}\text{bisimilar unfoldings into tree structures}\\ \text{preservation under bisimulation}\end{array}\right\} \Rightarrow \text{ tree model property}$$

tree model property

for all $\sim$-invariant logics $\mathsf{ML}, \ldots, \mathsf{L}_\mu, \ldots \mathsf{ML}_\infty$:
every satisfiable formula has a tree model

for $\approx$-invariant logics analogously: forest model property

of great importance: can employ good model theoretic and algorithmic properties of trees, MSO on trees, tree automata, . . . for robust decidability and complexity results for modal logics

finite (tree) model property

for basic modal logic ML (and some close relatives)
even get finite tree models, hence the

finite model property:

every satisfiable formula of ML has a finite (tree) model

ad-hoc method: for $\varphi \in$ ML of nesting depth ℓ,
truncate tree model at depth ℓ (preserving $\sim^\ell$)
and prune $\sim^\ell$-equivalent siblings (finite index!)

more generic method: passage to $\sim^\ell$-quotient of any model yields a finite model (usually not a tree model)

this generalises to extensions preserved under levels of $\approx$

capturing bisimulation-invariant Ptime

Ptime/$\sim$ $\left\{ \begin{array}{l} \text{the class of Ptime and } \sim\text{-closed} \\ \text{properties of finite structures} \end{array} \right.$

a semantic class

corr. to the undecidable class of Ptime Turing machines $\mathbb{M}$ that accept (encodings of) finite structures $\mathcal{A}, a$ and satisfy
$\mathcal{A}, a \sim \mathcal{B}, b \quad \Rightarrow \quad (\, \mathbb{M}[\mathcal{A}, a] = 1 \; \Leftrightarrow \; \mathbb{M}[\mathcal{B}, b] = 1 \,)$

capturing issue: **a logic for Ptime/$\sim$?**
does this semantic class admit
some syntactic representation?

yes, by straightforward reduction to **Immerman–Vardi** (O_ 96)

- use pre-processing $\mathcal{A} \longmapsto \mathcal{A}/\sim$ as a filter to enforce $\sim$-invariance
- quotients $\mathcal{A}/\sim$ carry canonical Ptime ordering of $\sim$-types ...
  $\rightarrow$ reduction to capturing Ptime over ordered finite structures

expressive completeness

...relative to first-order logic, a classical theme of FO model theory

FO/$\sim$ $\left\{\begin{array}{l}\text{the class of } \sim\text{-closed FO-properties of} \\ \text{(just finite, or all) relational structures}\end{array}\right.$

a semantic class

corresponding to the undecidable class of those $\varphi(x) \in$ FO
that satisfy $\mathcal{A}, a \sim \mathcal{B}, b \;\Rightarrow\; \big(\mathcal{A}, a \models \varphi \;\Leftrightarrow\; \mathcal{B}, b \models \varphi \big)$

classical 'preservation thms', too, respond to the quest for syntactic representation —mostly without asking the question

in this case, the answer to the unasked question is:

yes, FO/$\sim\ \equiv$ ML   classically, van Benthem

yes, FO/$\sim\ \equiv$ ML   in fmt, Rosen

expressive completeness: FO/$\sim$ $\equiv$ ML

it suffices to show:

$$\varphi(x) \in \mathrm{FO}_q/\!\sim \;\Rightarrow\; \varphi \in \mathrm{FO}/\!\sim^\ell$$
$$\text{for some } \ell = \ell(q) \quad (q = \mathrm{qr}(\varphi))$$

$\sim$-invariance implies $\sim^\ell$-invariance
a compactness property!

then $\varphi \equiv \varphi' \in \mathrm{ML}_\ell$, by Ehrenfeucht–Fraïssé:
finite index of $\sim^\ell$, ML_ℓ-definability of $\sim^\ell$-classes

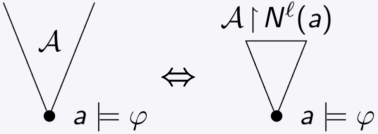
NB: two, a priori independent, readings: classical & fmt

expressive completeness: FO/$\sim$ $\equiv$ ML

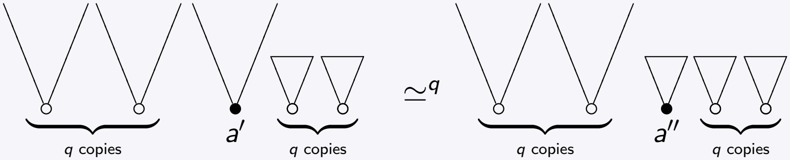
a simple, ad-hoc argument (with extra benefits)
using the locality of FO/$\sim$ & Ehrenfeucht–Fraïssé

$$\varphi(x) \in \mathrm{FO}_q/\!\sim \;\Rightarrow\; \varphi \in \mathrm{FO}/\!\sim^{\ell}$$
$$\text{for } \ell = 2^q - 1 \quad (q = \mathrm{qr}(\varphi))$$

show that



in q-round FO game:



expressive completeness

what is generic about the ad-hoc argument for FO/$\sim$?

- necessary & sufficient compactness property
  $\sim$-invariance $\Leftrightarrow$ ($\sim^{\ell}$-invariance for some ℓ)

- upgrading $\sim^{\ell(q)} \rightsquigarrow \equiv^{q}_{\mathrm{FO}}$

- FO-locality (Gaifman-locality)

what is not? **(e.g., compared to FO/$\approx$)**

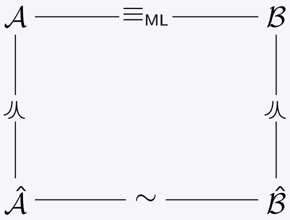
- locality around single distinguished vertex

want more uniform construction: $\approx$ coverings

expressive completeness: generic classical approach

$\sim$-invariance $\Rightarrow$ $\sim^{\ell}$-invariance for some ℓ

classical compactness argument allows upgrading along $\equiv_{\mathrm{FO}}$-axis through Hennessy–Milner property for ω-saturated structures

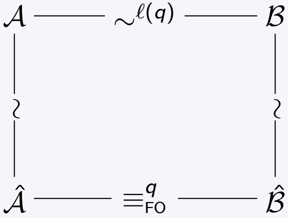


elegant and smooth, but no information regarding $\ell(q)$

expressive completeness: a constructive approach

$\sim$-invariance $\Rightarrow$ $\sim^{\ell}$-invariance for some ℓ

upgrading along $\sim$-axis — from $\sim^{\ell(q)}$ to $\simeq^{q}$ / $\equiv^{q}_{\mathrm{FO}}$
through bisimulation preserving model transformation (coverings)



more constructive, potentially suitable for fmt,
yielding information regarding $\ell(q)$

expressive completeness: a constructive approach

upgrading in

$$\begin{array}{ccc} \mathcal{A} & \text{———} \sim^{\ell(q)} \text{———} & \mathcal{B} \\ \wr & & \wr \\ \hat{\mathcal{A}} & \text{———} \equiv^{q}_{\mathsf{FO}} \text{———} & \hat{\mathcal{B}} \end{array}$$

requires (finite) model transformations $\mathcal{A}/\mathcal{B} \longmapsto \hat{\mathcal{A}}/\hat{\mathcal{B}}$

- compatible with $\sim/\approx$ (like $\approx$ coverings)
- suitable to eliminate all obstacles to $\simeq^{q}$ / $\equiv^{q}_{\mathsf{FO}}$ that are *not controlled* by any level of $\sim^{\ell}$

  esp., **short cycles** & **small multiplicities**

products and coverings

- products with reflexive cliques:
  boost multiplicities

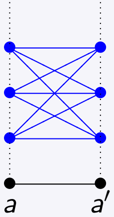
- products with generic graphs of large girth:
  avoid short cycles

products: direct synchronous products of $\mathcal{A} = (A, E)$

- with reflexive n-clique K_n:

  $\hat{\mathcal{A}} = (\hat{A}, \hat{E}) = \mathcal{A} \otimes K_n$
  $\hat{A} = A \times [n]$
  $\hat{E} = \{((a,i),(a',i')) \colon (a,a') \in E\}$

  projection homomorphism $\pi \colon (a,i) \mapsto a$



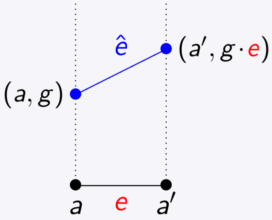
are bisimilar coverings (in the sense of $\approx$)

products and coverings

- products with reflexive cliques:
  boost multiplicities
- products with generic graphs of large girth:
  avoid short cycles

products: direct synchronous products of $\mathcal{A} = (A, E)$

- with Cayley graph $G = (G, (R_e)_{e \in E})$
  $\hat{\mathcal{A}} = (\hat{A}, \hat{E}) = \mathcal{A} \otimes G$
  $\hat{A} = A \times G$
  $\hat{E} = \{((a,g),(a', g \cdot e)) \colon e = (a,a') \in E\}$
  projection homomorphism $\pi \colon (a,g) \mapsto a$



are bisimilar coverings (in the sense of $\approx$)

III: the combinatorics of finite coverings

in this section:

- **graph coverings** (review)
- local acyclicity in finite direct products
  with Cayley graphs of large girth

- **hypergraph coverings** (new)
- degrees of acyclicity in hypergraphs
- acyclicity in finite reduced products
  with Cayley graphs of groupoids

graph coverings

w.l.o.g. consider directed loop-free graphs $\mathcal{A} = (A, E)$

definition: $\approx$-bisimilar coverings

$\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ a *covering* of $\mathcal{A} = (A, E)$ by $\hat{\mathcal{A}} = (\hat{A}, \hat{E})$:

(*forth*) $\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ homomorphism

(*back*) π lifts edges/paths from $a \in \mathcal{A}$ to any $\hat{a} \in \pi^{-1}(a)$

examples of simple/unbranched coverings:

- two-way tree (forest) unfoldings
- direct products of $\mathcal{A}$ with suitable graphs
  that are rich enough to simulate all $\mathcal{A}$-transitions
- especially: products with Cayley graphs generated by edge set E
  that serve as universal E-simulators

avoiding short cycles in finite coverings

NB: finite coverings of cyclic $\mathcal{A}$ must have cycles

N-acyclic coverings:
no (undirected) cycles of length up to N in covering

Cayley groups/graphs:

- group $G = (G, \cdot, 1)$ with generators $e \in E$
- associated Cayley graph has e-coloured edges from g to $g \cdot e$

highly symmetric, regular & homogeneous objects

Cayley groups of large girth (girth $> N$)**:**

no short generator cycles: $e_1 \cdot e_2 \cdots e_n \neq 1$ for $n \leqslant N$

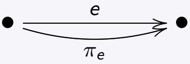
products $\mathcal{A} \otimes$ G with such G are N-acyclic coverings

Cayley graphs of large girth

no short generator cycles: $e_1 \cdot e_2 \cdots e_n \neq 1$ for small n

construction (after Biggs)

find G as subgroup $G = \langle \pi_e \colon e \in E \rangle \subseteq \mathrm{Sym}(V)$
generated by permutations π_e of
deterministically E-coloured graph $(V, (R_e))$



lemma

let $H = (V, (R_e))$ be deterministically E-coloured such that every colour sequence $w = e_1 \cdots e_n \in E^{\leqslant N}$ labels some path

$$v_0 \xrightarrow{e_1} v_1 \ \ \ldots \ \ v_{n-1} \xrightarrow{e_n} v_n \neq v_0 \quad \text{in } H;$$

then $\pi_{e_1} \cdots \pi_{e_n} \neq 1$ in $G \subseteq \mathrm{Sym}(V)$ and G has girth $> N$

locally acyclic graph coverings

thm (O_ 04)

every finite graph admits, for every $N \in \mathbb{N}$,
simple/unbranched N-acyclic finite coverings
by products with Cayley graphs of large girth

- **uniform construction, which preserves all symmetries**
- **adaptable to many special frame classes** → Dawar–O_ 05/09

construction idea for Cayley graphs extends to
stronger notions of acyclicity in groups and in groupoids
that are useful towards hypergraph constructions

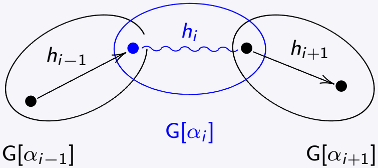
more than just large girth

much stronger notion of acyclicity in Cayley groups/graphs:
avoid not just short generator cycles but short coset cycles

coset cycles (in Cayley group G with generator set E)

$(g_i G[\alpha_i])_{i \leqslant n}$
cosets w.r.t. subgroups generated by $\alpha_i \subseteq E$
s.t. the transitions $h_i := g_i^{-1} \cdot g_{i+1} \in G[\alpha_i]$ satisfy:

$h_i \notin G[\alpha_i \cap \alpha_{i-1}] \cdot G[\alpha_i \cap \alpha_{i+1}]$



G is N-acyclic if it admits no coset cycles of length up to N

and such objects do exist

N-acyclic Cayley groups

thm (O_ 10)

for every finite set E and $N \in \mathbb{N}$ there are Cayley groups with generators $e \in E$ that admit no coset cycles of length up to N

inductively interleave

- **amalgamation of chains of Cayley graphs of small subgroups**
- **group action on deterministically coloured graphs**

to avoid coset cycles in increasing no.s of generators

from graphs to hypergraphs

hypergraphs: structures $\mathcal{A} = (A, S)$ with vertex set A,
and set of hyperedges $S \subseteq \mathcal{P}(A)$

idea: clusters and their link structure

example: hypergraph of guarded subsets
of a relational structure $\mathcal{A} = (A, \mathbf{R}^{\mathcal{A}})$

$$\mathbf{H(\mathcal{A}) = (A, S[\mathcal{A}])}$$

with hyperedges generated by subsets $[\mathbf{a}] \subseteq A$ for $\mathbf{a} \in R^{\mathcal{A}}$, $R \in \mathbf{R}$
closed under subsets & singleton sets

relational structure	=	**hypergraph link structure (topology)**
		+ local relational content

$\rightsquigarrow$ hypergraph bisimulations/coverings take care of
the combinatorial part of guarded bisimulations/coverings

hypergraphs

hypergraph terminology

- $\mathbf{H} = (\mathbf{A}, \mathbf{S})$, $S \subseteq \mathcal{P}(A)$ the set of hyperedges
- $\mathbf{G(H)} = (\mathbf{A}, \mathbf{E})$, associated Gaifman graph
  hyperedges $\rightsquigarrow$ cliques
- $\mathbf{G}(\mathcal{A}) = \mathbf{G}(\mathbf{H}(\mathcal{A}))$, the Gaifman graph of $\mathcal{A}$

issues:

- **degrees of acyclicity** and their algorithmic and model-theoretic relevance ($\rightarrow$ guarded logics, part IV)
- **hypergraph coverings:** reproduce link structure locally; smooth out global link structure (e.g., regarding cycles)

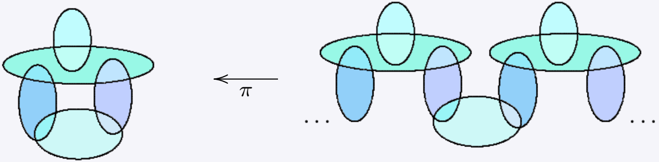
hypergraph coverings

definition: bisimilar coverings

$\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ a *covering* of $\mathcal{A} = (A, S)$ by $\hat{\mathcal{A}} = (\hat{A}, \hat{S})$:

(*forth*) $\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ homomorphism
i.e., $\pi \restriction \hat{s} \colon \hat{s} \rightarrow \pi(\hat{s}) = s \in S$ bijective for all $\hat{s} \in \hat{S}$

(*back*) π lifts overlaps $s \cap s' \neq \emptyset$ from $\mathcal{A}$ to any $\hat{s} \in \hat{S}$ above s



examples of natural hypergraph coverings:

- tree (forest) unfoldings
- reduced products with suitable groups/groupoids ($\rightarrow$ below)

degrees of hypergraph acyclicity

hypergraph acyclicity; 3 equivalent definitions:

- tree-decomposable with hyperedges as bags

  **associate hyperedges of $\mathcal{A}$ with nodes of tree T s.t.
  every a $\in$ A is represented in connected subgraph of T**

- decomposable through elementary deletion steps (Graham)

  – **delete simply covered vertices**
  – **delete subset-hyperedges**

- **conformality and chordality** (of associated Gaifman graph)

  – **no bad cliques in Gaifman graph**
  – **no bad cycles in Gaifman graph**

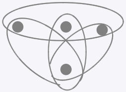
hypergraph terminology

for hypergraph $\mathrm{H} = (A, S)$ and associated Gaifman graph

$\mathrm{G}(\mathrm{H}) = (A, E) = \bigcup_{s \in S} K[s]$ (a clique for each $s \in S$)

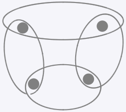
- **conformality:** every clique in G(H) is contained in some $s \in S$

no



- **chordality:** every cycle of length > 3 in G(H) has a chord

no



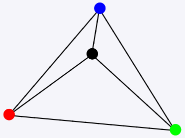
N-acyclicity = N-conformality + N-chordality:

acyclicity of induced sub-configurations of size up to N

example: the combinatorial challenge

the facets of the 3-simplex/tetrahedron

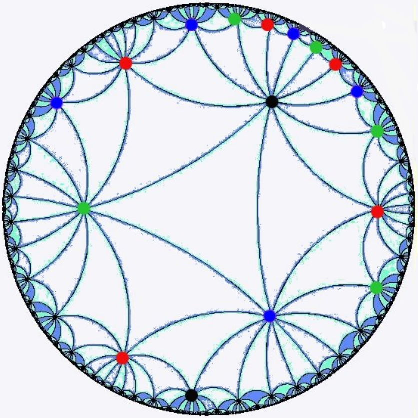
uniform width 3 hypergraph on 4 vertices

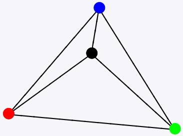


- chordal but not conformal
- finite coverings cannot be 1-locally acyclic
- admits locally finite coverings without short chordless cycles
- also admits simple finite 5-acyclic covering in which every induced sub-configuration on up to 5 vertices is acyclic

example: the combinatorial challenge

a locally finite covering **of the tetrahedron**





conformal; shortest chordless cycles have length 12

here by regular triangulation of the hyperbolic plane

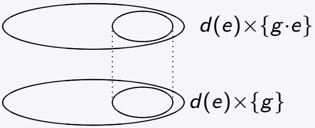
reduced products with Cayley groups/groupoids

plain reduced product $\mathcal{A} \otimes \mathsf{G}$

between hypergraph $\mathcal{A} = (A, S)$ and group/groupoid
with generators $e \in E$ associated with subsets $d(e) \in S\!\downarrow$

$$\mathcal{A} \otimes \mathbf{G} : \begin{cases} \text{quotient } (\mathcal{A} \times \mathsf{G})/\!\approx \\ (a, g) \approx (a, g') \quad \text{if} \quad g^{-1} \cdot g' \in \mathsf{G}[\alpha_a] \\ \qquad\qquad\qquad\qquad\quad \text{for } \alpha_a = \{e \in E \colon a \in d(e)\} \end{cases}$$

intuition: e-transitions in G glue layers of $\mathcal{A} \times \mathsf{G}$ through identification in d(e)



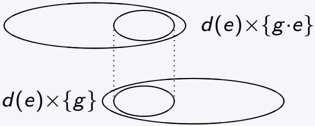
reduced products with Cayley groups/groupoids

unfolded reduced product $\mathcal{A}^\iota \otimes \mathbf{G}$

of incidence representation of $\mathcal{A} = (A, S)$ and group/groupoid with generators $e = (s, s') \in E$ associated with subsets $d(e) = s \cap s'$ for $s \cap s' \neq \emptyset$

$$\mathcal{A}^\iota \otimes \mathbf{G} : \begin{cases} \text{quotient } (\dot{\bigcup} S \times \mathrm{G})/{\approx} \\ (a, s, g) \approx (a, s', g \cdot e) \;\text{ if } g^{-1} \cdot g' \in \mathrm{G}[\alpha_a] \\ \alpha_a = \{e = (s, s') \in E \colon a \in s \cap s'\} \end{cases}$$

intuition: e-transitions in G for $\mathrm{e = (s, s')}$ glue copies of s and s′ in appropriate layers



new methods: Cayley graphs of groupoids

theorem

- plain reduced products with N-acyclic Cayley groups G preserve N-acyclicity of $\mathcal{A}$

$\rightsquigarrow$ **local–global construction of finite N-acyclic coverings from locally finite N-acyclic coverings** (O_ 10)

- unfolded reduced products with N-acyclic Cayley groupoids G produce N-acyclic coverings of $\mathcal{A}$

$\rightsquigarrow$ **direct construction of finite N-acyclic coverings** (new)

. . . and N-acyclic groups/groupoids can be constructed by very similar group action & amalgamation ideas

further (new) results

reduced product constructions with N-acyclic groupoids yield

generic solutions for finite closures/realisations of

- abstract specifications of local overlap patterns
- abstract specifications of complete GF-types $\rightarrow$ part IV
- extension properties for partial isomorphisms
  (in the sense of Hrushovski/Herwig/Lascar) $\rightarrow$ part IV

these highly regular & symmetric constructions
are compatible with automorphisms of the given data
(preserve symmetries of the sepecification)

. . . **and why groupoids?**

groupoids vs. groups

groupoids: think of 'many-sorted' groups with partial (sort-sensitive) operation

$$\mathbf{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}, {}^{-1}\big)$$

with operation $G_{st} \times G_{tu} \stackrel{\cdot}{\longrightarrow} G_{su}$

examples: bijective morphisms in a category; change of co-ordinates

why groupoids are more suitable in hypergraph constructions

- transitions between hyperedges
  behave like local changes of co-ordinates
  – with non-trivial compositions
- (reduced) products with groupoids can offer
  just the right transitions at the right place

. . . **unlike the graph/group situation**

IV: finite model theory of guarded logics

in this section:

- guarded logics and guarded bisimulation
- generalised tree model property
- finite model properties
- descriptive complexity
- expressive completeness

the guarded fragment GF

Andréka–van Benthem–Németi 98

model-theoretic motivation: reflection on ML $\subseteq$ FO
from graph-like structures to general relational format

key idea: relativise quantification to guarded clusters

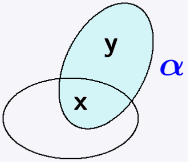
hypergraph of guarded subsets $H(\mathcal{A}) = (A, S[\mathcal{A}])$
generated by $[\mathbf{a}]$ for $\mathbf{a} \in R^{\mathcal{A}}$

guarded quantification:

$$\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})\big)$$

$$\forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big)$$

guard atom α: $\mathrm{free}(\varphi) \subseteq \mathrm{var}(\alpha)$



quantification relativised to guarded tuples

GF and guarded bisimulation

guarded bisimulation $\quad \mathcal{A}, \mathbf{a} \sim_{\mathrm{g}} \mathcal{A}', \mathbf{a}'$ **and** $\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}}^{\ell} \mathcal{A}', \mathbf{a}'$

- bisimulation of hypergraphs of guarded subsets
  that locally respects relations
- FO pebble game with guarded pebble configurations

  two equivalent views (Grädel–Hirsch–O_ 02)

the guarded Ehrenfeucht–Fraïssé thm

$\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}}^{\ell} \mathcal{A}', \mathbf{a}' \quad \Leftrightarrow \quad \mathcal{A}, \mathbf{a} \equiv_{\mathrm{GF}}^{\ell} \mathcal{A}', \mathbf{a}'$ $\qquad$ (GF_{ℓ}-equiv./depth ℓ)

the guarded Karp thm

$\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}} \mathcal{A}', \mathbf{a}' \quad \Leftrightarrow \quad \mathcal{A}, \mathbf{a} \equiv_{\mathrm{GF}}^{\infty} \mathcal{A}', \mathbf{a}'$ $\qquad$ (inf. equiv. in GF_{∞})

GF and guarded bisimulation/coverings

in striking analogy with modal model theory, based on
invariance/preservation under guarded bisimulation:

- **generalised tree model property**
  tree/forest unfoldings (Grädel 99):
  acyclic hypergraph coverings

- **finite model properties (and decidability)**
  via Herwig extensions (Grädel 99)
  succinct hypergraph coverings (Bárány–Gottlob–O_ 10)

- **capturing result for $\sim_{\mathrm{g}}$-invariant Ptime**
  succinct hypergraph coverings (Bárány–Gottlob–O_ 10)

- **classical/fmt expressive completeness results**
  compactness&saturation/upgrading in coverings
  (Andréka–van Benthem–Németi 98/O_ 10)

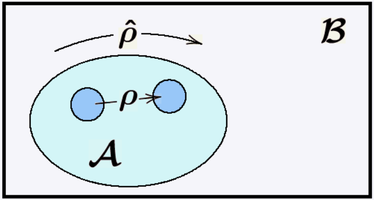
finite model properties

finite models from Herwig extensions (Grädel 99)

from infinite $\mathcal{A}^\infty \models \varphi$ obtain finite model as
Herwig-extension $\mathcal{B} \supseteq \mathcal{A}$ of sufficiently rich finite $\mathcal{A} \subseteq \mathcal{A}^\infty$

Hrushovski–Herwig–Lascar EPPA:

for finite $\mathcal{A}$ find finite extension $\mathcal{B} \supseteq \mathcal{A}$ that extends
every partial isomorphism of $\mathcal{A}$ to an automorphism of $\mathcal{B}$



w.l.o.g. $\mathbf{R}^{\mathcal{A}}$ generates $\mathbf{R}^{\mathcal{B}}$

if $\mathcal{A}$ represents $\mathcal{A}^\infty/\sim^\ell_{\mathrm{g}}$
then $\mathcal{B} \sim^\ell_{\mathrm{g}} \mathcal{A}^\infty$

more on EPPA & fmp for GF

Herwig–Lascar EPPA

within classes $\mathcal{C}$ defined in terms of
finitely many forbidden homomorphisms:

> if $\mathcal{A}$ has an infinite EPPA-extension $\mathcal{A} \subseteq \mathcal{B}^{\infty} \in \mathcal{C}$,
> then there is a finite EPPA-extension $\mathcal{A} \subseteq \mathcal{B}^{\mathrm{fin}} \in \mathcal{C}$

corollary

fmp for GF in restriction to any class $\mathcal{C}$ defined in terms of
finitely many forbidden homomorphisms

first obtained (with feasible size bounds) in Bárány–Gottlob–O_ 10
using succinct weakly *N*-acyclic covers
$\rightsquigarrow$ 'Rosati-covers' & 'finite controllability' of UCQ/GF

more on hypergraph constructions & EPPA

new application of reduced products w.r.t. N-acyclic groupoids:

- **new combinatorial proof of Herwig–Lascar EPPA theorem** based on finite, symmetric realisations of overlap specifications between isomorphic copies of $\mathcal{A}$

related task: model (re-)construction from abstract specification of complete GF-types

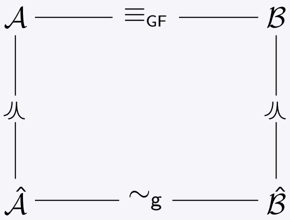
- Bárány–Gottlob–O_ 10: good bounds, unclear symmetries $\rightsquigarrow$ capturing bisimulation-invariant Ptime
- new groupoidal constructions: generic & fully symmetric, no feasible bounds (?)

expressive completeness: $\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$

crux (as in modal case): compactness property

$$\boxed{\varphi \in \mathrm{FO}\ \sim_{\mathrm{g}}\text{-invariant} \ \Rightarrow\ \sim_{\mathrm{g}}^{\ell}\text{-invariance for some } \ell}$$

- classical compactness argument allows upgrading along $\equiv_{\mathrm{FO}}$-axis, by use of ω-saturated elementary extensions

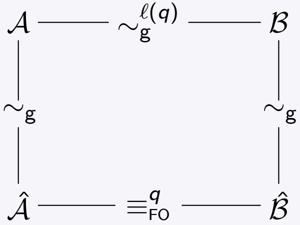


expressive completeness: FO/$\sim_{\mathrm{g}}$ $\equiv$ GF

crux (as in modal case): compactness property

$$\varphi \in \mathrm{FO}\ \sim_{\mathrm{g}}\text{-invariant} \;\Rightarrow\; \sim_{\mathrm{g}}^{\ell}\text{-invariance for some } \ell$$

- constructive upgrading along $\sim_{\mathrm{g}}$-axis
  uses rich N-acyclic (finite) coverings



beyond GF: guarded negation

Bárány–ten Cate–Segoufin 11

idea: consider benign nature of GF (and ML) in light of *restricted negation* rather than restricted quantification

- start from existential FO (UCQ)
- allow negation just on formulae with explicitly guarded free variables

GF $\subseteq$ GNF $\subseteq$ FO

appropriate notion of bisimulation combines (local) homomorphisms with (guarded) b&f

allows to lift many results from GF

largely by non-trivial reductions to GF over classes with forbidden homomorphisms

summary: bisimulation & link structure

combinatorics, discrete geometry/topology

analogies and generalisations: modal $\rightsquigarrow$ guarded

discrete mathematics: graphs $\rightsquigarrow$ hypergraphs

databases: transition systems $\rightsquigarrow$ relational databases

logic/model theory: modal $\rightsquigarrow$ guarded logics

e.g., tree-decompositions and tree unfoldings
& finite coverings with control over cycles

how far do the analogies carry?

summary: how far do bisimulation analogies carry?

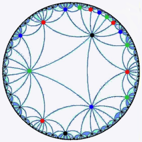
- infinite tree unfoldings as fully acyclic coverings:
  a complete analogy, good for most classical purposes

- finite coverings meet different combinatorial challenges
  w.r.t. control of cycles and local-global-distinctions

- gain considerable extensions of the analogies between
  graphs/hypergraphs & modal/guarded logics

- especially through new hypergraph constructions
  via reduced products with suitable groupoids

the end



some pointers

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