The Freedoms of Guarded Bisimulation

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questions

- **what is bisimulation good for?**
- **how does bisimulation generalise to hypergraphs?**
- **what is guarded bisimulation good for?**
- **which features of bisimulation generalise?**
- **what are the combinatorial challenges?**

bisimulation – the quintessential back & forth

state-transition systems

transition systems: coloured directed graphs

Kripke structures: possible worlds, accessibility relations

temporal structures: states, flow of time

epistemic structures: knowledge states, uncertainty equivalences

game graphs: positions and possible moves

notions of behaviour

sequences of transitions (between observable states)

interactive behaviour: challenge/response instead of traces

embeddable trees of action sequences (up to multiplicities)

bisimulation classes

the bisimulation game

back & forth in transition systems

with binary (transition) relations $\mathbf{R} = (R_1, \dots)$
and unary (state) predicates $\mathbf{P} = (P_1, \dots)$

two players on **two structures:**

$\mathcal{A} = (A, (\mathbf{R}^{\mathcal{A}}), (\mathbf{P}^{\mathcal{A}}))$ vs. $\mathcal{A}' = (A', (\mathbf{R}^{\mathcal{A}'}), (\mathbf{P}^{\mathcal{A}'}))$

game positions:

$(a, a') \in A \times A'$ pebbles on a in $\mathcal{A}$ and on a' in $\mathcal{A}'$

single round, challenge/response:

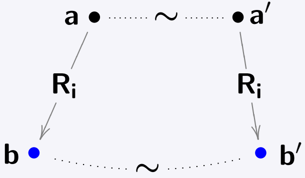
player **I** makes a transition from a or from a'
player **II** needs to match this transition on opposite side

the bisimulation game

back & forth

single round:

challenge/response



winning/losing:

- player **II** needs to maintain local equivalence between states
- player **I** or **II** lose when stuck

winning strategies and back & forth systems

winning strategies for player II:

$\mathcal{A}, \mathbf{a} \sim^{\ell} \mathcal{A}', \mathbf{a}'$ player **II** has winning strategy in ℓ-round game from position $(a; a')$

$\mathcal{A}, \mathbf{a} \sim \mathcal{A}', \mathbf{a}'$ player **II** has winning strategy in unbounded game from position (a, a')

winning regions as relations:

$$Z_\ell := \{(a, a') \in A \times A' : \mathcal{A}, a \sim^{\ell} \mathcal{A}', a' \}$$

$$Z_\infty := \{(a, a') \in A \times A' : \mathcal{A}, a \sim \mathcal{A}', a' \}$$

(nondet.) winning strategies as back & forth systems:

graded by no. of remaining rounds for $\sim^{\ell}$ / flat for $\sim$

modal logic

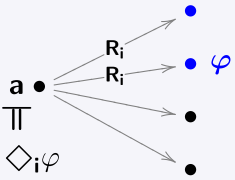
basic modal logic ML

with binary (transition) relations $\mathbf{R} = (R_1, \ldots)$ $\rightsquigarrow$ modalities $\Diamond_i / \Box_i$
and unary (state) predicates $\mathbf{P} = (P_1, \ldots)$ $\rightsquigarrow$ basic propositions p_i

atomic formulae: $\bot, \top$ and p_i
closure under booleans and
modal quantification:

$$\Diamond_i \varphi \equiv \exists y \big(R_i xy \wedge \varphi(y) \big)$$
$$\Box_i \varphi \equiv \forall y \big(R_i xy \rightarrow \varphi(y) \big)$$

relativised FO quant.



example: $\Diamond_1 \Box_2 \Diamond_1 \Box_2 p$

bisimulation equivalence – modal Ehrenfeucht–Fraïssé

the modal Ehrenfeucht–Fraïssé thm

t.f.a.e. for any $\mathcal{A}, a$ and $\mathcal{A}', a'$:

(i) $\mathcal{A}, a \sim^{\ell} \mathcal{A}', a'$

(ii) $\mathcal{A}, a \equiv^{\ell}_{\mathrm{ML}} \mathcal{A}', a'$ (equivalence w.r.t. ML up to depth ℓ)

consequences:

- invariance/preservation: ML^{ℓ} preserved under $\sim^{\ell}$
  ML preserved under $\sim$
- tree model property of modal logics (!!)

bisimulation equivalence – modal Ehrenfeucht–Fraïssé

the modal Karp thm:

t.f.a.e. for any $\mathcal{A}, a$ and $\mathcal{A}', a'$:

(i) $\mathcal{A}, a \sim \mathcal{A}', a'$

(ii) $\mathcal{A}, a \equiv^{\infty}_{\mathrm{ML}} \mathcal{A}', a'$ (equivalence w.r.t. infinitary ML)

consequences:

- invariance/preservation: ML_{∞} preserved under $\sim$
- Hennessy–Milner thm: $\equiv_{\mathrm{ML}}$ coincides with $\sim / \equiv^{\infty}_{\mathrm{ML}}$ on 'saturated' models
- classical proof of van Benthem's characterisation of ML

expressive completeness

modal characterisation thm (van Benthem)

$$\boxed{\mathbf{FO}/{\sim} \;\equiv\; \mathbf{ML}}$$

for $\varphi(x) \in$ FO: φ preserved under $\sim$ $\;\Leftrightarrow\;$ $\varphi \equiv \varphi'$ with $\varphi' \in$ ML

ML captures precisely those FO properties
that are bisimulation-invariant

finite model theory (fmt) analogue (Rosen)

$$\boxed{\mathbf{FO}/{\sim} \;\equiv\; \mathbf{ML}\ \text{(fmt)}}$$

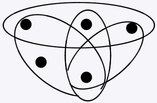
for $\varphi(x) \in$ FO: φ preserved under $\sim$ over finite structures $\;\Leftrightarrow\;$ $\varphi \equiv_{\mathrm{fin}} \varphi'$ with $\varphi' \in$ ML

observable configurations in relational structures

examples:

- tuples in relational database,
- clusters of variables in CSP and conjunctive queries,
- higher-arity roles (as in description logics)

so as to model: clustering of states
non-binary link structures
restrictions on (simultaneous) access



guardedness

observable configurations in relational structures

examples:

- tuples in relational database,
- clusters of variables in CSP and conjunctive queries,
- higher-arity roles (as in description logics)

the essence of the generalisation

from graphs to hypergraphs

transition systems/graphs	$\longrightarrow$	relational structures/hypergraphs
modal logic	$\longrightarrow$	guarded logic
bisimulation	$\longrightarrow$	**guarded bisimulation**

the hypergraph of guarded subsets

of a relational structure $\mathcal{A} = (A, \mathbf{R}^{\mathcal{A}})$:

$$\mathbf{H(\mathcal{A}) = (A, S[\mathcal{A}])}$$

with hyperedges $[\mathbf{a}] \subseteq A$ for every $\mathbf{a} \in R^{\mathcal{A}}$, $R \in \mathbf{R}$

$[\mathbf{a}] = \{a_1, \dots, a_r\}$ if $\mathbf{a} = (a_1, \dots, a_r)$
$+$ closure under subsets and singleton sets

general terminology:

- hypergraph $\mathbf{H = (A, S)}$
  $S \subseteq \mathcal{P}(A)$ the set of hyperedges $s \in S$
- induced graph: $\mathbf{G(H) = (A, E)}$
  $E = \{(a, a') \colon a \neq a', \{a, a'\} \subseteq s$ for some $s \in S\}$

  $\mathbf{G(H(\mathcal{A}))}$: **Gaifman graph** $\mathbf{G(\mathcal{A})}$

the guarded fragment

GF: quantification relativised to guarded tuples

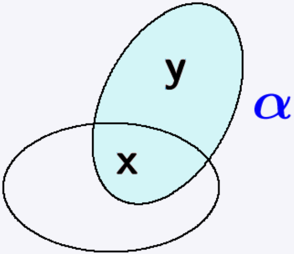
atomic formulae of FO
closure under booleans and
guarded quantification

$$\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})\big)$$

$$\forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big)$$

with guard atom α s.t.

$$\mathrm{free}(\varphi) \subseteq \mathrm{free}(\alpha) = \mathrm{var}(\alpha)$$



example: $\forall \mathbf{x}(R\mathbf{x} \rightarrow \exists \mathbf{y}(W\mathbf{xy} \wedge \neg Q\mathbf{y}))$

ML $\subsetneq$ GF $\subsetneq$ FO

the natural extension of modal pattern to arbitrary relations

the guarded fragment

key properties of GF

- finite model property
- decidable for SAT = FINSAT
- bounded tree width property (and more)
- preservation/characterisation (guarded bisimulation)

in striking analogy with ML

well-behaved extensions:

CGF: allow Gaifman cliques as guards

μ**GF**: GF + least fixed points (Grädel–Walukiewicz 99)

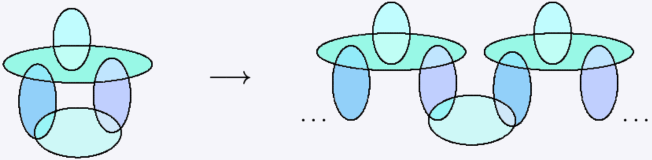
GNF: guarded negation fragment (Barany–ten Cate–Segoufin 11)

from graphs to hypergraphs

hypergraph tree unfoldings

based on tree unfolding of
intersection graph between hyperedges
for identifications of nodes in overlaps

result: a tree-decomposable hypergraph $\hat{H} \sim H$



$\longrightarrow$ generalised tree-model property (Grädel)

hypergraph bisimulation & guarded bisimulation

hypergraph bisimulation $\mathbf{H, s \sim H', s'}$ **and** $\mathbf{H, s \sim^{\ell} H', s'}$

idea: bisimulation of the intersection graphs
moves between hyperedges respecting the overlap

position in game on $\mathbf{H = (A, S)}$ vs. $\mathbf{H' = (A', S')}$:

bijections $s \leftrightarrow s'$, $s \in S, s' \in S'$

single round, challenge/response:

player **I** selects $t \in S$ or $t' \in S'$
player **II** needs to complete to new bijection $t \leftrightarrow t'$
compatible with $s \leftrightarrow s'$ on $s \cap t$ (on $s' \cap t'$)

II loses when stuck

hypergraph bisimulation & guarded bisimulation

hypergraph bisimulation $\mathbf{H}, \mathbf{s} \sim \mathbf{H}', \mathbf{s}'$ and $\mathbf{H}, \mathbf{s} \sim^{\ell} \mathbf{H}', \mathbf{s}'$

single round, challenge/response:

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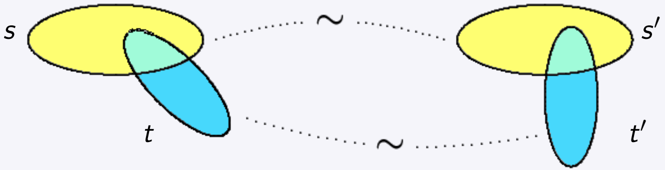


challenge/response
move from • to •

hypergraph bisimulation & guarded bisimulation

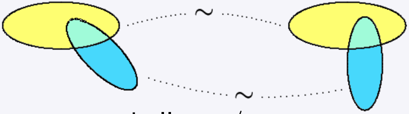
guarded bisimulation $\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}} \mathcal{A}', \mathbf{a}'$ **and** $\mathcal{A}, \mathbf{a} \sim_{\mathrm{g}}^{\ell} \mathcal{A}', \mathbf{a}'$

idea 1: bisimulation of hypergraphs of guarded subsets that locally respects relations

idea 2: pebble game with guarded pebble configurations

the two are equivalent

both captured by a bisimulation game on associated transition system of guarded tuples (Grädel–Hirsch–O_)



challenge/response

guarded bisimulation – guarded Ehrenfeucht–Fraïssé

the guarded Ehrenfeucht–Fraïssé thm

t.f.a.e. for any $\mathcal{A}, \mathbf{a}$ and $\mathcal{A}', \mathbf{a}'$:

(i) $\mathcal{A}, \mathbf{a} \;\sim_{\mathrm{g}}^{\ell}\; \mathcal{A}', \mathbf{a}'$

(ii) $\mathcal{A}, \mathbf{a} \equiv_{\mathrm{GF}}^{\ell} \mathcal{A}', \mathbf{a}'$ (equivalent w.r.t. GF up to depth ℓ)

... with Karp-style extension relating $\sim_{\mathrm{g}}$ and $\equiv_{\mathrm{GF}}^{\infty}$

consequences:

- invariance/preservation: GF$^{\ell}$ preserved under $\sim_{\mathrm{g}}^{\ell}$
  GF preserved under $\sim_{\mathrm{g}}$
- generalised tree model property of guarded logics (Grädel)
- characterisation thm, classical (Andreka–van Benthem–Nemeti)

guarded and modal, one more example

guarded characterisation thm (Andreka–van Benthem–Nemeti)

$$\boxed{\mathbf{FO}/\!\sim_{\mathrm{g}} \;\equiv\; \mathbf{GF}}$$

for $\varphi(x) \in$ FO: φ preserved under $\sim_{\mathrm{g}}$ $\Leftrightarrow$ $\varphi \equiv \varphi'$ with $\varphi' \in$ GF

GF captures precisely those FO properties
that are guarded bisimulation-invariant

remark: fmt version open until recently (more below)

$\mathrm{GSO}/\!\sim_{\mathrm{g}} \equiv \mu\mathrm{GF}$ (Grädel–Hirsch-O_)
$\mathrm{MSO}/\!\sim \;\equiv\; \mathrm{L}_\mu$ (Janin–Walukiewicz)
} both open in fmt

further themes:

(1) tree-like models: **acyclicity and its finite approximations**

(2) finite model properties

(3) expressive completeness in fmt

(4) bisimulation quotients, canonisation and capturing

hypergraph acyclicity

acyclicity of $\mathbf{H} = (\mathbf{A}, \mathbf{S})$ three equivalent characterisations:

- **H** admits reduction $\mathbf{H} \rightsquigarrow \emptyset$

  via decomposition steps: $\begin{cases} \text{delete } a \text{ if } a \in s \text{ for single } s \in S \\ \text{delete } s \text{ if } s \subsetneq s' \in S \end{cases}$

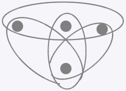
- **H** has tree decomposition $\delta \colon \mathcal{T} \to S$

- **H** is **conformal** & **chordal**

conformality:

every clique in $G(H)$ guarded

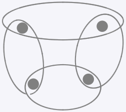
no



chordality:

every cycle of length $\geqslant 4$ has a chord

no



bisimilar covers

hypergraph cover:

$\boldsymbol{\pi} : \hat{\mathbf{H}} \longrightarrow \mathbf{H}$

hypergraph homomorphism
inducing hypergraph bisimulation of bijections $\big(\hat{s} \leftrightarrow \pi(\hat{s})\big)$

local bijections with *back*-property w.r.t. overlap pattern

guarded cover:

$\boldsymbol{\pi} : \hat{\mathcal{A}} \longrightarrow \mathcal{A}$

relational homomorphism
inducing guarded bisimulation of local isomorphisms $\big(\hat{s} \leftrightarrow \pi(\hat{s})\big)$
between guarded substructures

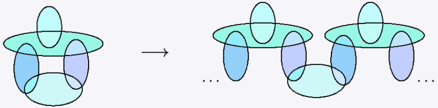
local isomorphisms with hypergraph cover property,
hypergraph cover through local isomorphisms

covers: unclutter locally—preserve link structure

first examples:

hypergraph tree unfoldings guarded tree unfoldings	are	acyclic hypergraph covers acyclic guarded covers

just as plain tree unfoldings of transition systems
are bisimilar graph covers by trees



fact: tree unfoldings of cyclic structures are infinite

how much acyclicity is possible in finite covers?

the combinatorial challenge

(1) acyclicity in finite covers

conformal covers (Hodkinson–O_ 03)

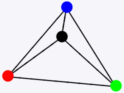
- every finite hypergraph admits a cover
  by a finite conformal hypergraph

- every finite relational structure admits a guarded cover
  by a finite structure which is conformal

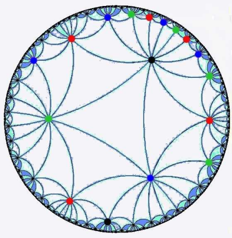
method: suitable local restriction of 'free' covers
that cover s by graphs of functions $\rho\colon s \rightarrow \{1, \ldots, k\}$

(1) acyclicity in finite covers

even 1-local chordality cannot
generally be obtained in finite covers



locally finite cover of
tetrahedron on ●, ●, ●, ●



relaxation:

N-chordality
require chordality only for short cycles

(1) acyclicity in finite covers

(a) weak N-chordality of a cover $\pi\colon \hat{\mathsf{H}} \longrightarrow \mathsf{H}$

short chordless cycles in $\mathsf{G}(\hat{\mathsf{H}})$ acquire chords in projection to H

weakly N-acyclic covers (Barany–Gottlob–O_ 10)

every finite hypergraph admits finite conformal and weakly N-chordal covers

. . . and analogue for relational structures

method: quotients of term-based structures inspired by Rosati's chase

$\longrightarrow$ essentially optimal complexity

(1) acyclicity in finite covers

(b) full N-chordality

no short chordless cycles G(H)

fully N-acyclic covers (O_ 10)

every finite hypergraph admits covers by
finite conformal and fully N-chordal hypergraphs

. . . and analogue for relational structures

ingredients: generalisation of Cayley groups of large girth

$+$ a local-to-global construction
and glueing to mend defects

$\longrightarrow$ maximal acyclicity, but no feasible bounds

(2) finite model properties

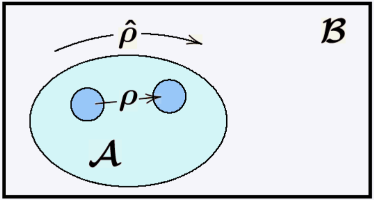
finite model property for GF

Grädel's proof based on Herwig's EPPA

extension properties for partial isomorphisms (EPPA)

Hrushovski, Herwig, Herwig–Lascar

for finite $\mathcal{A}$ and partial iso $\rho \in \mathrm{Part}(\mathcal{A}, \mathcal{A})$,
can find finite $\mathcal{B} \supseteq \mathcal{A}$ with $\rho \subseteq \hat{\rho} \in \mathrm{Aut}(\mathcal{B})$

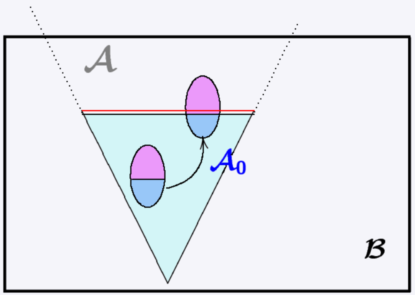


w.l.o.g. $S[\mathcal{B}] = \langle S[\mathcal{A}] \rangle^{\mathrm{Aut}(\mathcal{B})}$
no 'new' guarded sets

finite model property for GF

Grädel's proof based on Herwig's EPPA

after relational Skolemisation ($\varphi \in \mathrm{GF} \rightsquigarrow \varphi' \in \forall\exists\mathrm{GF}$):
use EPPA to obtain finite model as finite closure
of finite substructure of infinite model
w.r.t. guarded $\forall\exists$-requirements



$\longrightarrow$ cover constructions
for optimal bounds

(2) extensions of fmp for GF

conformal structures and CGF

existence of finite conformal covers (Hodkinson–O_ 03)
$\Rightarrow$ **fmp(GF) in the class of all conformal structures**

applications:

- proof of **fmp**(**CGF**) using the fact that

  CGF $\equiv$ GF   over conformal structures

  where clique guarded = guarded

- extension of EPPA

  e.g. to K_n-free graphs

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further application: extension of EPPA e.g. to K_n-free graphs

(2) extensions of fmp for GF

similarly, finite weakly N-acyclic covers give:

fmp with forbidden homomorphisms (Barany–Gottlob–O_ 10)

let $\mathcal{C}$ be the class of all $\mathcal{A}$ without homomorphisms $\mathcal{B} \xrightarrow{hom} \mathcal{A}$
for a given finite list of finite $\mathcal{B}$

fmp(GF) over $\mathcal{C}$:

> if $\varphi \in$ GF has any model in $\mathcal{C}$,
> then φ has a finite model in $\mathcal{C}$

application: finite controllability of unions of conjunctive queries w.r.t. guarded constraints: $\varphi \models q \;\Leftrightarrow\; \varphi \models_{\mathrm{fin}} q$

optimal size and complexity bounds
essentially as good as for GF alone!

(2) extensions of fmp for GF

similarly, finite fully N-acyclic covers give:

fmp with forbidden cyclic configurations (O_ 10)

let $\mathcal{C}$ be the class of all $\mathcal{A}$ without substructures $\mathcal{B} \subseteq \mathcal{A}$
for a given finite list of finite cyclic $\mathcal{B}$

fmp(GF) over $\mathcal{C}$:

if $\varphi \in$ GF has any model in $\mathcal{C}$,
then φ has a finite model in $\mathcal{C}$

(3) expressive completeness in fmt

$\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$ in fmp (O_ 10)

fmt analogue of Andreka–van Benthem–Nemeti characterisation with a radically different proof of expressive completeness:

> GF expresses, *over finite structures,* every FO property that is invariant under $\sim_{\mathrm{g}}$ *on finite structures*

crux (modulo Ehrenfeucht–Fraïssé):

φ invariant under $\sim_{\mathrm{g}}$ in all finite structures $\Rightarrow$ φ invariant under some $\sim_{\mathrm{g}}^{\ell}$ in all finite structures

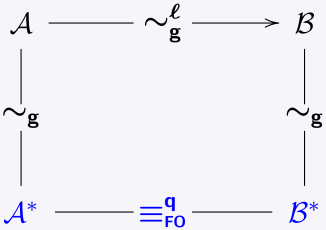
proof uses finite N-acyclic covers
to control FO^q-type by GF^ℓ-types

(3) expressive completeness in fmt

crux (modulo Ehrenfeucht–Fraïssé):

φ invariant under $\sim_{\mathrm{g}}$ in all finite structures $\Rightarrow$ φ invariant under some $\sim_{\mathrm{g}}^{\ell}$ in all finite structures

upgrading $\sim_{\mathbf{g}}^{\ell}$ ($\equiv_{\mathsf{GF}}^{\ell}$) to $\equiv_{\mathsf{FO}}^{\mathsf{q}}$:



sufficiently rich & acyclic covers

(4) bisimulation quotients, canonisation, capturing

from finite structure $\mathcal{A}$ abstract in Ptime

I($\mathcal{A}$) bisimulation quotient of the game graph for the guarded bisimulation game on $\mathcal{A}$

succinct description of $\mathcal{A}/\!\sim_{\mathrm{g}}$

$\mathbf{I}(\mathcal{A}) = \mathbf{I}(\mathcal{A}')$ **iff** $\mathcal{A} \sim_{\mathrm{g}} \mathcal{A}'$ **complete invariant**

Ptime canonisation w.r.t. $\sim_{\mathrm{g}}$ (Barany–Gottlob–O_ 10)

weakly acyclic covers serve to construct from I a canonical realiser:

$\hat{\mathcal{A}} \sim_{\mathrm{g}} \mathcal{A}$ **and** $\hat{\mathcal{A}} = \hat{\mathcal{A}}'$ **iff** $\mathcal{A} \sim_{\mathrm{g}} \mathcal{A}'$ **canonisation**

applications:
- capturing result for $\mathrm{Ptime}/\!\sim_{\mathrm{g}}$
- optimal bounds for small models of GF and CGF

summary

- **guarded bisimulation is for relational structures**
  (and hypergraph bismulation is for hypergraphs)

  what bisimulation is for graph-like structures

- **degrees of hypergraph acyclicity in finite covers**
  much harder to achieve than in the graph case

  . . . but of similar importance and success

- **hypergraphs/relational structures of qualified acyclicity**
  have interesting structure theory (e.g., bdd convex hulls)

- **combinatorics of finite hypergraph covers**
  remains a challenge (e.g., compatibility with automorphisms)