Finite Global Realisations of Local Overlap Specifications

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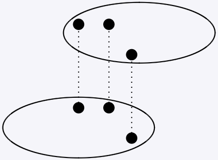
PTM & DMV

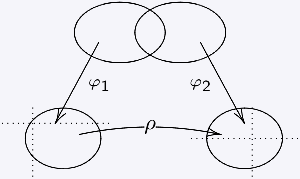
Groupoids, Hypergraphs, and Symmetries in Finite Models

examples of local views & specifications

- **exploded view of a hypergraph**



- **coordinate charts for manifolds**



- **decomposition and synthesis** of graphs, hypergraphs, ...
- **implicit specifications** of bisimulation types $p \rightarrow \Diamond q$
- **extension properties** $\forall \mathbf{x}\big(\theta(\mathbf{x}) \rightarrow \exists \mathbf{y}\theta'(\mathbf{xy})\big)$

local —— global

atlas of local maps with changes of coordinates	**manifolds**
distinguished substructures with overlap specifications	**relational structures**
hyperedges with overlap specifications	**hypergraphs**
partial isomorphisms with composition in overlaps	**automorphism groups**

local specifications? **global realisations?**

the role of groupoids or inverse semigroups

two ‘equivalent’ algebraic formats for
inclusion and composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

groups capture global symmetries:
symmetry groups as automorphism groups
within the full symmetric group

groupoids & inverse semigroups capture local symmetries:
inverse semigroups of partial isomorphisms
within the full symmetric inverse semigroup

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedeges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

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intersection graph:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') \colon s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges $s \in S$

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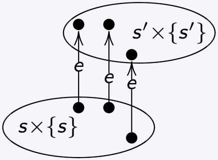
exploded view:

the disjoint union of the hyperedges $s \in S$
with partial bijections ρ_e for $e = (s, s') \in E$
specifying identifications in overlaps



$\rightsquigarrow$ **format of local overlap specifications**

overview

- motivation: local & global views
- **(I) specification & realisation of overlap patterns**
- **(II) reduced products with groupoids (core results)**
- **(III) from local to global symmetries**

(I) abstract specification & realisation

incidence pattern $\mathrm{I} = (S, (E[s,s'])_{s,s' \in S})$

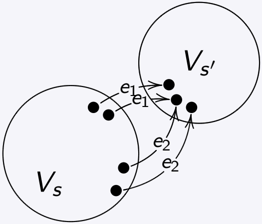
multi-graph with vertices $s \in S$ (sorts)
directed edges $e \in E[s,s']$ from s to s' with $e^{-1} \in E[s',s]$

- **fixed bisimulation type for pairwise overlaps**

I-graph $\mathrm{H} = \big(V, (V_s)_{s \in S}, (\rho_e)_{e \in E}\big)$

vertex set V partitioned into sorts V_s for $s \in S$
ρ_e a partial* bijection between V_s and $V_{s'}$ for $e \in E[s,s']$

- **an exploded view of the desired pairwise overlaps**

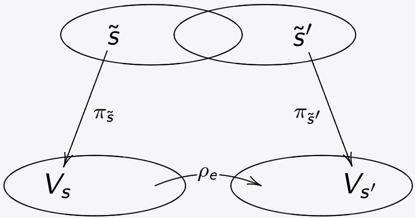


realisation of $\mathrm{H} = (\mathbf{V}, (\mathbf{V_s}), (\rho_e))$

hypergraph $\mathcal{A} = (A, \tilde{S})$ with projection $\pi \colon \tilde{S} \longrightarrow S$
and an atlas of bijections $\pi_{\tilde{s}} \colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

- **all specified overlaps are realised:**
  for $e \in E[s, s']$, ρ_e is realised at every $\tilde{s} \in \pi^{-1}(s)$
  by an actual overlap with some $\tilde{s}' \in \pi^{-1}(s')$

- **no further, incidental overlaps occur:**
  all actual overlaps of $\mathrm{I}(\mathcal{A})$ are induced by
  compositions ρ_w of partial bijections ρ_e in H



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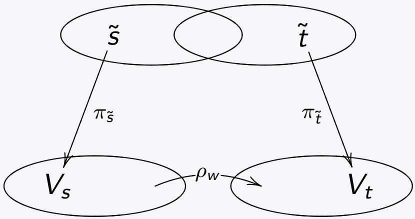

realisations vs. exploded views

the exploded view of hypergraph $\mathcal{A} = (A, S)$
is an I-graph $\mathrm{H}(\mathcal{A})$ w.r.t. $\mathrm{I}(\mathcal{A}) = (S, E)$

$\mathcal{A}$ is a realisation of $\mathrm{H}(\mathcal{A})$ obtained as a quotient $\mathrm{H}(\mathcal{A})/\approx$
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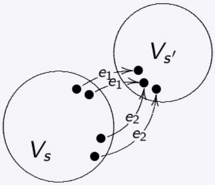
in general, the natural quotient $\mathrm{H}/\approx$
may fail to be a realisation of the I-graph H:



$\approx$ may even collapse individual $\mathrm{V_s}$

idea: try local unfolding in products of H with . . . ?

(II) reduced products with groupoids

for fixed incidence pattern $\mathrm{I} = (S, E)$:

I-groupoid $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

- **I-groupoids come with Cayley graphs that are I-graphs**

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reduced products as candidate realisations:

$\rightsquigarrow$ $\mathrm{H} \times \mathbb{G}$ natural direct product (of I-graphs)

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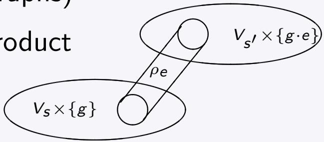

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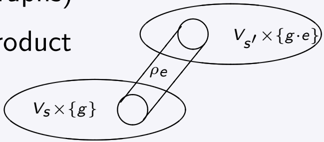
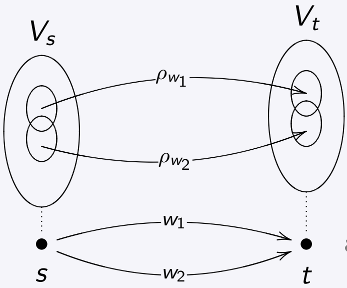


a realisation of H?

- **for $\mathrm{H} = \mathrm{H}(\mathcal{A})$, $\mathrm{H} \otimes \mathbb{G}$ is a covering of the hypergraph $\mathcal{A}$**

obstructions to simple realisations

- $\mathbb{H}$ may fail to be *coherent*:
  a lack of path-independence in $\mathbb{H}$, with conflicting identifications collapsing individual V_s

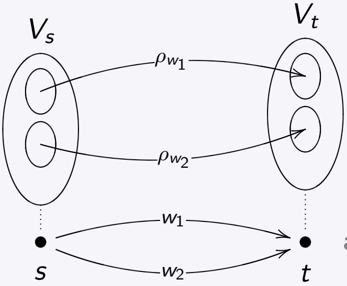


a bad cycle in $\mathbb{G}$?

can be overcome by relatively simple pre-processing: almost w.l.o.g. assume coherence in this sense

obstructions to simple realisations

- $\mathbb{H}$ and $\mathbb{G}$ may fail to be *confluent* in the product: another lack of path-independence, with potentially conflicting identifications at the relational level

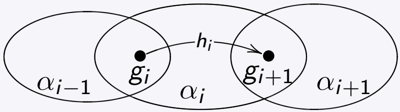


a bad cycle in $\mathbb{G}$?

$\rightsquigarrow$ **need substantial acyclicity conditions on $\mathbb{G}$**

the right notion of acyclicity

- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



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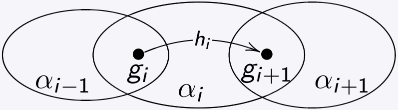
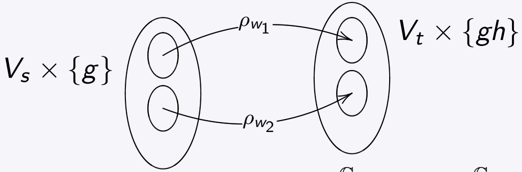
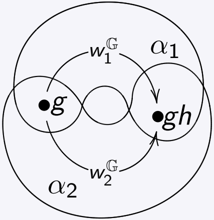


- in particular, need to avoid certain coset cycles of length 2



$$h_1 = w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}} = h_2^{-1}$$



any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

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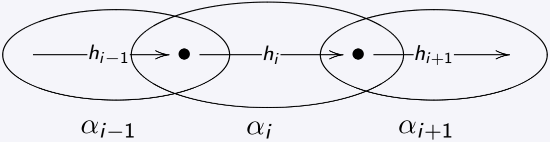
inductive construction:

- generate I-groupoids from (semi)group action on I-graphs
- eliminate short coset cycles by the use of amalgamation chains of $\mathrm{I}[\alpha]$-graphs that unfold short cosets cycles



cf. constructions of acyclic Cayley graphs (Alon, Biggs)
lifted to intricate adaptation for coset cycles
$\rightarrow$ O_10 (JACM 13) for groups

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H}$ (an I-graph), obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

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symmetric realisations

corollary

every finite hypergraph admits, for $\mathsf{N} \in \mathbb{N}$, finite coverings that

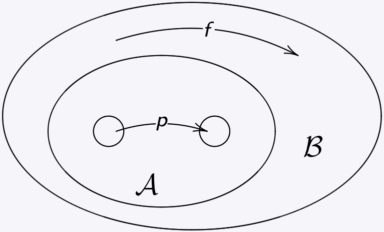
- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

(III) from local to global symmetries

EPPA: extension property for partial automorphisms;
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$



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theorem (Herwig–Lascar 00)

---

same, as a *finite model property* over any class $\mathcal{C}$
defined by finitely many forbidden homomorphisms

from local to global symmetries: 3 proof sketches

**EPPA: extension property for partial automorphisms;
how to extend local symmetries to global symmetries**

(1) the Herwig–Lascar proof 'from the book':
EPPA in the class of all finite graphs (Hrushovski's thm)

(2) example/realisations towards EPPA:
extension of a single partial isomorphism of $\mathcal{A}$
to an automorphism of some $\mathcal{A}' \supseteq \mathcal{A}$

(3) new proof of full Herwig–Lascar EPPA:
through highly symmetric & acyclic realisations of
an overlap specification within a restricted class $\mathcal{C}$

(1) a proof 'from the book' (Herwig–Lascar)

for finite simple graph (V, E) find extension that lifts all partial isomorphisms of (V, E) to automorphisms

(i) pass to dual picture by representing (V, E) as the intersection graph of $(E, \{E[v] \subseteq E \colon v \in V\})$
with edges represented as size 1 intersections

(ii) extend E to $X \supseteq E$ by dummy elements for small $E[v]$ so that (V, E) becomes a substructure of the full intersection graph of some $(X, [X]^n)$

(iii) check that every partial isomorphism of (V, E) is induced by a partial bijection of $(X, [X]^n)$, hence extends to an automorphism of the intersection graph of $(X, [X]^n)$

(2) example/realisations towards EPPA

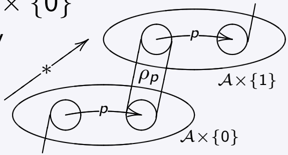
extension of a single partial isomorphism p of $\mathcal{A} = (A, R)$
to an automorphism of some $\mathcal{A}' \supseteq \mathcal{A}$

(i) **from a free infinite realisation:**
find simple infinite extension/realisation in reduced product $(\mathcal{A} \times \mathbb{Z})/\approx$ where $\approx$ is induced by the partial bijections

$$\begin{array}{rcl} \rho_p^{i,i-1} \colon \mathcal{A} \times \{i\} & \longrightarrow & \mathcal{A} \times \{i-1\} \\ (a, i) & \longmapsto & (p(a), i-1) \end{array}$$

$\mathcal{A} \simeq \mathcal{A} \times \{0\} \subseteq (\mathcal{A} \times \mathbb{Z})/\approx$ and p in $\mathcal{A} \times \{0\}$ extends to the automorphism induced by

$$* \colon (a, i) \longmapsto (a, i+1)$$



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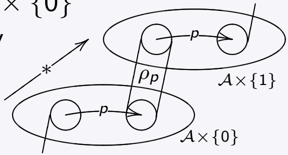


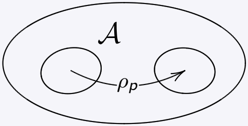
(ii) **to a cyclic finite realisation:**
for any $n \geqslant 3$ such that $p^n = \mathrm{id}_s$
a local identity on suitable s ⊆ dom(p) ∩ image(p)
$(\mathcal{A} \times \mathbb{Z}_n)/\!\approx$ similarly yields a finite extension/realisation

(3) new proof of full Herwig–Lascar EPPA

through highly symmetric & acyclic realisations of an overlap specification for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I}(\mathcal{A}, \mathbf{P})$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H}(\mathcal{A}, \mathbf{P})$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!



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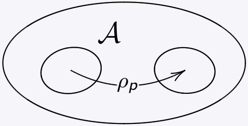

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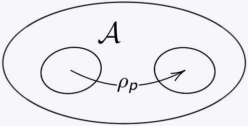

(iii) **symmetric realisations of $\mathbf{H(\mathcal{A}, P)}$ are EPPA extensions !**

(iv) **N-acyclic EPPA extensions are N-free:**
admit N-local homomorphisms into every (finite or infinite) EPPA extension due to their N-local tree-decomposability

applications in algorithmic model theory

- **finite model properties & finite controllability for guarded logics and constraints**

  using finite coverings of controlled acyclicity and/or Herwig–Lascar extension properties

- **characterisation theorems** (fmt) **for guarded logics and relatives**

  using finite coverings of controlled acyclicity

$\longrightarrow$ **Bárány–Gottlob–O** (LICS10&LMCS14)
Bárány–ten Cate–O (VLDB12)
O (LICS10&JACM13)
O (APAL13)
O (LICS13&arXiv14)

summary

- a generic construction of
  highly acyclic finite groupoids
- a universal & generic route to the synthesis of
  finite realisations (and coverings) in reduced products
- symmetry and acyclicity of realisations supports
  extensions of local to global symmetry

⟶ **Finite Groupoids, Finite Coverings & Symmetries in Finite Structures** (arXiv 2014)