Celebrating Erich
Berlin 2018

ABC
A amalgamation
B bisimulation
C cycles

piecing together fitting parts
games & strategies
rounding it out
content

- **A is for amalgamation**
  from small- to large-scale local acyclicity
  or from local to global consistency
  or from local to global symmetries

- **B is for bisimilar coverings**
  for the local unravelling of cycles
  in graphs or hypergraphs

- **C is for cycles**
  in graphs and hypergraphs or
  in groups and groupoids
  with new constructions
  to locally avoid cycles

really, A is for B, B is for C, C is for A, . . .
amalgamation?

identification in local overlaps, as in

- Fraïssé limits

- extending local to global symmetries
  Hrushovski/Herwig/Herwig–Lascar EPPA

- Erich’s elegant proof of fmp for GF
bisimulation?

the quintessential b&f equivalence:

\(\sim\) bisimulation for transition systems/Kripke structures
b&f between graph-like structures

\(\sim_g\) guarded bisimulation for relational structures
b&f between hypergraph-like structures
walks and cycles in graphs & hypergraphs?

walk/path in graph

walk/path in hypergraph

... with different potential for clutter
acyclicity = tree-likeness

in graphs: obvious

in hypergraphs: maybe less so
(I) cycles in graphs/cycles in hypergraphs

cycles in graphs (and graph-like relational structures)
– not governed by bisimulation
– avoidable in (infinite) bisimilar tree unfoldings
– acyclicity preserved in weak substructures

cycles in hypergraphs (and relational structures)
– not governed by (guarded) bisimulation
– avoidable in (infinite, guarded) bisimilar unfoldings: generalised tree model property for GF (Grädel 99)
– acyclicity not preserved in weak substructures (!)
aside 1: what even is a hypergraph cycle?

- hypergraph acyclicity  \((\alpha\text{-acyclicity, Fagin et al)}\)
- tree-decomposability of hypergraphs  \((\text{Graham)}\)
- conformality & chordality  \((\text{Berge, Fagin et al)}\)

all equivalent, as notions of hypergraph acyclicity
but what are the relevant cycles?

Julian Bitterlich: new notion of *hypergraph cycle*,
characterising hypergraph acyclicity
aside 1: what even is a hypergraph cycle?

a naive cycle is a **B-cycle** if no three overlaps covered together!

Bitterlich 2018: hypergraph acyclicity $\Leftrightarrow$ no such cycles with nice structure theory to match
aside 2: tree models & finite models (ML)

• ML has tree model property
  \sim\text{-invariance, tree unfolding}

• ML has finite model property
  \sim^\ell\text{-invariance, }\sim^\ell\text{-quotients}

• ML has finite tree model property
  \sim^\ell\text{-invariance, }\ell\text{-tree unfolding & }\sim^\ell\text{-pruning}
aside 2: tree-like models vs. finite models (GF)

- GF has generalised tree model property (Grädel 99)
  \( \sim_g \)-invariance, hypergraph tree unfolding
- GF has finite model property (Grädel 99)
  \( \sim_\ell \)-invariance does not support quotients (!)

Erich’s use of Herwig’s EPPA thm:
truncated hypergraph tree unfolding, completed in Herwig EPPA extension

- GF has locally tree-like finite models in combination with (O_12)
aside 3: uses of locally acyclic (finite) models

finite and algorithmic model theory, esp. modal and guarded: bisimulation invariance in non-elementary settings

- van Benthem–Rosen thms over natural frame classes and also for substantial extensions (like GF, ML[CK])
  O_04, Dawar–O_09, O_13, Ciardelli–O_18, O_12, Canavoi–O_17

- analysis of guarded negation and guarded constraints
  Bárany–ten Cate–O_12, Bárany–Gottlob–O_14

→ The freedoms of (guarded) bisimulation  Grädel–O_14
(II) cycles in groups and groupoids

Cayley group
with generators $e \in E$

\[ \mathcal{G} = (G, \cdot, 1, (e)_{e \in E}) \quad \quad \quad w \in E^* \mapsto w^\mathcal{G} \in G \]

Cayley graph

\[ \mathcal{G} = (G, (R_e)_{e \in E}), R_e = \{(g, ge): g \in G\} \quad \quad \quad w \in E^*: \text{walks in } \mathcal{G} \]

first shot: cycles = generator cycles (= graph cycles)

\[ \leadsto \text{finite groups of large girth} \]
aside: no short generator cycles = large girth

t.f.a.e.:
- $w^G \neq 1$ for (reduced) generator words $|w| \leq 2\ell + 1$
- Cayley graph $G$ has girth $\geq 2\ell + 1$
- is $\ell$-locally acyclic
- $G \upharpoonright \mathcal{N}(1)$ a tree

\[
\begin{align*}
&\quad a \quad b \\
&ca \quad \text{a} \quad c \\
&\quad b \quad c \\
&\quad b \quad \text{a} \\
&\quad c \\
&\quad b \quad a \\
&\quad bc
\end{align*}
\]

e.g. with involutive generators $a, b, c$
cycles in groups and groupoids

Cayley group
with (involutive) generators $e \in E$ ($e^{-1} = e$)
$$\mathbb{G} = (G, \cdot, 1, (e)_{e \in E})$$
$$w \in E^* \mapsto w^\mathbb{G} \in G$$

Cayley graph
$$\mathbb{G} = (G, (R_e)_{e \in E}), R_e = \{(g, ge) : g \in G\}$$
$$w \in E^* : \text{walks in } \mathbb{G}$$

first shot: cycles = generator cycles (= graph cycles)

second shot: cycles = coset cycles (= hypergraph cycles)

cyclic configurations formed by cosets
$$g\langle \alpha \rangle^\mathbb{G} = g\{w^\mathbb{G} : w \in \alpha^*\} \text{ for generator subsets } \alpha \subseteq E$$
no short coset cycles – much more than large girth

forbid short cyclic configurations

of overlapping hyperedges without internal shortcuts

NB: walk in $g\langle \alpha \rangle \iff$ single coset-step in $\alpha$
aside: e.g. coset 2-acyclicity

no coset 2-cycles:

\[ g' \in g\langle \alpha \rangle^G \cap g\langle \beta \rangle^G \]

implies \( g' \in \langle \alpha \cap \beta \rangle^G \)
cycles in groups and groupoids

motivation for coset acyclicity:

transitions in graph-like structures \(\text{vertex-to-vertex}\) are memory-less

transitions in hypergraphs \(\text{hyperedge-to-hyperedge}\) preserve elements in overlap

\(\sim\) different generators (overlaps) fix same element and form cosets of related group elements

coset cycles of \(G\) can be seen as hypergraph cycles in

Cayley hypergraph

a hypergraph dual of \(G\) with cosets as elements
... and analogously for groupoids

**groupoid:** many-sorted with partial, sort-dependent operation

instead of arbitrary generator words $w \in E^*$: walks $w$ in fixed directed graph $\mathbb{I} = (S, E)$

$\leadsto$ elements of sorts $G[s, s']$ for $s, s' \in S$

concatenation/products in matching sites/sorts

$$\mathbb{G} = (G, (G[s, s']), \cdot, (1_s), (e)) \quad \quad w \in \mathbb{I}[s, s] \mapsto w^G \in G[s, s']$$

**motivation:** distinct extensions/operations in different sites

e.g. in hypergraph coverings
Cayley groups and groupoids (summary)

Cayley group/groupoid

with generators \( e \in E \) over \( \mathbb{I} = (S, E) \) \( w \in \mathbb{I}^* \rightleftharpoons w^G \in G \)

with Cayley graph

\[ G = (G, (R_e)_{e \in E}), \quad R_e = \{(g, ge) : g \in G\} \]

\( w \in \mathbb{I}^* : \) walks in \( G \)

and Cayley hypergraph of cosets

\[ g \langle \alpha \rangle^G : \text{for } g \in G, \alpha \subseteq E \]

\( w \in \mathbb{I}^*(\alpha) : \) walks preserving \( \bigcap_\alpha \cdot \)

\( \ldots \) hopefully avoiding short generator or even coset cycles
(III) finite cover constructions

use (reduced) products with the right kind of group(oid):

- tree unfolding of graph \((V, E)\) as a weak subgraph of direct product with free Cayley group with generator set \(E\)

- finite graph coverings with local acyclicity inherited from Cayley group of large girth

- (finite) graph coverings with local acyclicity even w.r.t. some specific transitive closures from Cayley group w/o short coset cycles

- finite hypergraph coverings with local acyclicity inherited from Cayley groupoid w/o short coset cycles
direct and reduced products: basic idea

finite graph coverings in direct product with Cayley group: large girth $\leadsto$ local acyclicity

finite hypergraph coverings in reduced product with Cayley groupoid: coset-acyclicity $\leadsto$ local acyclicity

$e = \{s, s'\}$
(IV) local acyclicity in Cayley group(oid)s

constructions:

• first shot: groups of large girth (Biggs 89)

• second shot: coset acyclicity in group(oid)s

• new: coset acyclicity for groupoids in groups
Cayley groups of large girth (first shot): basic idea

Biggs’ construction, example:

involutive generators $a, b, c$
find $G$ of girth $> 9$

need $w^G \neq 1$ for $|w| \leq 9$
e.g. for $w = ababcabc$

in $\{a, b, c\}$-coloured tree of depth $d = 2$
look at permutations $\pi_e$ for $e \in \{a, b, c\}$,
where $\pi_e$ swaps vertices within $e$-edges

\[ G := \langle \pi_a, \pi_b, \pi_c \rangle \subseteq \text{Sym}(V) \text{ has girth } \geq 4d + 2 \]
coset acyclicity (second shot) in Cayley groups

Biggs’ idea interleaved with amalgamation:

$\pi_e, \pi_w$ act on amalgamated cosets (unfolded coset cycles)
e.g., to force $\pi_w \neq \text{id}$ in $\text{Sym}(V)$ for $w = u_1 u_2 u_3$, $u_i \in \alpha_i^*$

challenge: upgraded $G \subseteq \text{Sym}(V)$ must not mess up these $\langle \alpha_i \rangle$
by induction w.r.t. size can avoid new $\beta \cap \alpha_i$
acylicity in Cayley groupoids: old and new

• (2012 ...): adaptation of amalgamation idea for groups to more challenging setting of groupoids

• new: ramified acyclicity condition for groups can cover interpretation of groupoid patterns in group instead of all walks \( w \in \alpha^* \) in cosets \( \langle \alpha \rangle \):
  • walks w.r.t. given groupoid pattern \( \mathbb{I} = (S, E) \) (reg. constr.)
  • focus on corresponding weak subgraphs of the cosets \( \langle \alpha \rangle \)
  • eliminate small cyclic configurations among those
interpretation of groupoids inside groups

- can interpret directed (groupoid) edges of $\mathbb{I} = (S, E)$ as paths of undirected (involutive, group) edges in $E''$

- extract $\mathbb{I}$-groupoid $\mathcal{G}$ from $E''$-group $\mathcal{G}''$

get coset acyclicity in groupoid $\mathcal{G}$ from $\mathbb{I}$-coset acyclicity in group $\mathcal{G}''$
theorems

get constructions of finite group(oid)s for given $E$ and $N$ that are
- coset $N$-acyclic
- compatible with given finite (hyper)graph
- generic in respecting all symmetries of the given data

major applications:
- finite graph and hypergraph coverings that are locally acyclic
- extensions of local to global symmetries: strengthening Hrushovski, Herwig–Lascar EPPA thms
- Cayley structures as $\sim$-generic common knowledge models: characterisation thms (with Felix Canavoi)
- constructions in semi-group theory: proof of Henckell–Rhodes conjecture (Julian Bitterlich)
references


