Extensions of Partial Automorphisms, Hypergraphs and Guarded Logics

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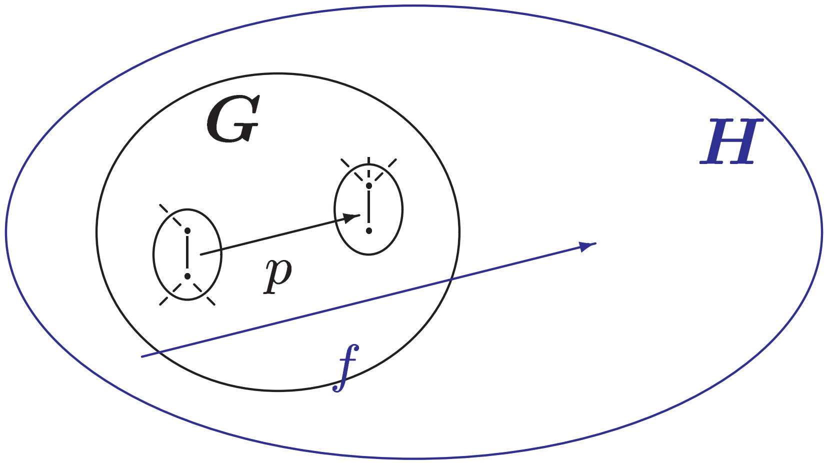
→ forthcoming BSL paper on joint work with **Ian Hodkinson**

Extension of Partial Automorphisms

Truss, Hrushovski, Herwig, Lascar, . . .

extension theorem basic version, for finite undirected graphs

given $\boxed{\begin{array}{l}\textbf{finite } G\\ p \in \mathrm{Part}(G)\end{array}}$ find $\boxed{\begin{array}{l}\textbf{finite } H \supseteq G\\ p \subseteq f \in \mathrm{Aut}(H)\end{array}}$



finiteness!

with variations: extend one p
extend all p
preserve other features

generic automorphisms and SIP for Fraïssé limits

e.g., rather than the class of finite graphs G
consider expansions $(G, f_1, \ldots)$, $f_i \in \mathrm{Aut}(G)$

JEP and **AP** guarantee limit structure

$$(\boldsymbol{G_0}, \boldsymbol{F_1}, \ldots)$$

random graph expanded by generic automorphisms
universally embedding for $(G, p_1, \ldots)$, $p_i \in \mathrm{Part}(G)$

EPPA

briefest history of results

(1)	**Truss 92**	graphs, one p
(2)	**Hrushovski 92**	graphs, all p
(3)	**Herwig 95**	arbitrary relational structures
(4)	**Herwig 98**	omitting certain configurations e.g.: K_n-free graphs Henson digraphs
(5)	**Herwig/Lascar 00**	$\rightarrow$ theory of free groups $+$ combinatorial proofs of (2),(3)

two tracks $\begin{cases} \textbf{group theoretic} \\ \textbf{combinatorial} \end{cases}$ (5) covers (2) and (3), not (4)

omitting configurations: forbidden homomorphic images of packed structures (Gaifman cliques)

Lascar's simple combinatorial proof (for graphs)

$G = (V, E)$ finite, undirected $\qquad n := \max_v d_E[v]$

extend to $\hat{E} := E \,\dot\cup\, \{$ dummies $\}$ such that $d_{\hat{E}}[v] = n$ for all v

homogeneous hull

$$\boxed{\begin{array}{l} \boldsymbol{H = (U, I)} \\ U = [\hat{E}]^n, \ n\text{-element subsets} \\ I = \{(s, s') \colon s \cap s' \neq \emptyset\} \end{array}}$$

with embedding
$$\begin{array}{rcl} \rho \colon G & \xrightarrow{\simeq} & H \\ v & \longmapsto & \{e \in \hat{E} \colon v \text{ incident with } e\} \end{array}$$

$$\Rightarrow \quad |\rho(u) \cap \rho(v)| \leqslant 1 \text{ for all } u, v \in G$$

$$(u, v) = e \in E \quad \text{iff} \quad \rho(u) \cap \rho(v) = \{e\}$$

- $\mathrm{Aut}(\boldsymbol{H}) = \mathrm{Sym}(\hat{\boldsymbol{E}})$
- $\boldsymbol{H} \supseteq \boldsymbol{G}$ **has extensions for all** $p \in \mathrm{Part}(G)$

what's new here

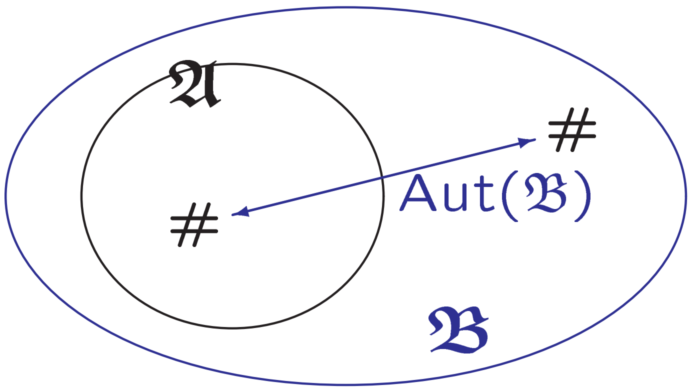
combinatorial approach

links with
- hypergraph theory
- fmt of guarded logics

omitting configurations (in the sense of $\simeq$ rather than hom)

linked to Gaifman cliques (cf. packed)

keep extension $\mathfrak{B} \supseteq \mathfrak{A}$ tight w.r.t. such:



a class of configurations #

$\mathfrak{B} \supseteq \mathfrak{A}$ **tight w.r.t. #** if
$\mathfrak{B}$ realises no "new" #

$\Rightarrow$ **EPPA tight w.r.t. all "packed configurations"**

relational structures and their Gaifman graphs

relational structure $\mathfrak{A} = (A, R^{\mathfrak{A}}, \ldots)$

write $[\boldsymbol{a}]$ for the *set* of components of tuple $\boldsymbol{a}$

$$\boxed{\begin{array}{l} G(\mathfrak{A}) = (A, E) \\ \qquad E = \{(a, a') \colon a \neq a' \in [\boldsymbol{a}], \boldsymbol{a} \in R^{\mathfrak{A}}\} \end{array}}$$

Gaifman graph

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guarded and clique guarded subsets/configurations

$s \subseteq \mathfrak{A}$ **guarded**: $\quad |s| = 1$ or $s = [\boldsymbol{a}]$, $\boldsymbol{a} \in R^{\mathfrak{A}}$

$$\left.\begin{array}{l} s \subseteq \mathfrak{A} \textbf{ clique guarded} \\ \mathfrak{A} \restriction s \textbf{ packed} \end{array}\right\} \quad G(\mathfrak{A}) \restriction s \text{ a clique}$$

NB: triangles are clique guarded/packed, k-cycles for $k > 3$ are not (in graphs)

hypergraphs / terminology

hypergraph $H = (A, S), S \subseteq 2^A$ with **hyperedges** $s \in S$

$G(H) = (A, E)$
$E = \bigcup_{s \in S} K[s]$ $\quad$ $K[s]$: clique on s

associated (Gaifman) graph

for relational $\mathfrak{A}$, in particular:

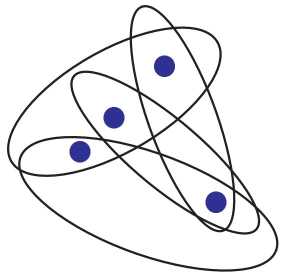
$H(\mathfrak{A}) = (A, S)$
$S = \{\{a\} : a \in A\} \cup \{[\boldsymbol{a}] : \boldsymbol{a}$ in $R^{\mathfrak{A}}\}$

hypergraph of guarded sets

NB: $G(\mathfrak{A}) = G(H(\mathfrak{A}))$

hyperedges of H give rise to a cliques in $G(H)$, but not every clique in $G(H)$ is covered by a hyperedge

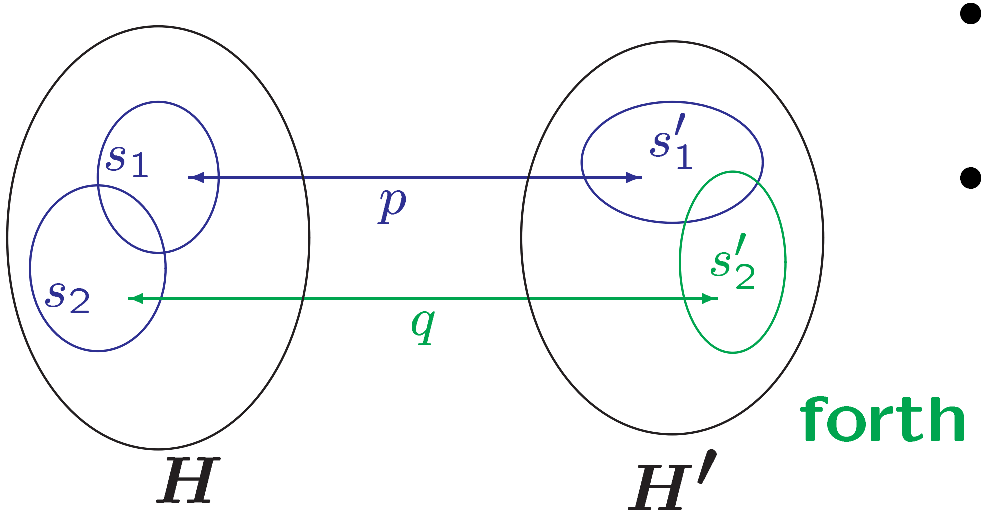
unguarded cliques in $G(H)$



back and forth between hypergraphs

hypergraph bisimulation

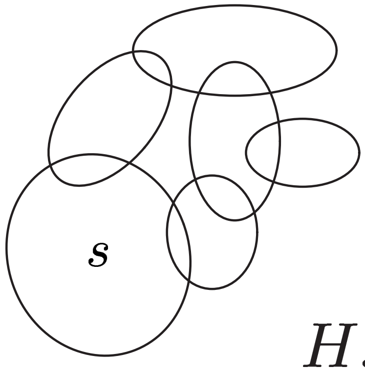
$$Z\colon H \approx H'$$



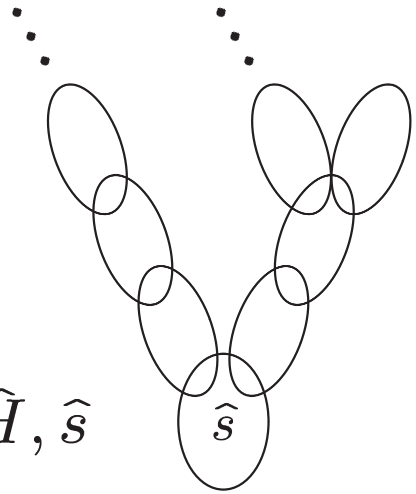
- $\emptyset \neq Z \subseteq \mathsf{Part}(A, A')$
- local bijections between hyperedges
- back & forth respecting overlaps

associated unravellings

bisimilar companions
built from tree of paths
of overlapping hyperedges



$$H, s \;\approx\; \widehat{H}, \widehat{s}$$



$\hat{H}$ is an acyclic hypergraph

tree decompositions of $G(\hat{H})$

$G(\widehat{H})$ has tree width max$|s| - 1$

acyclicity of hypergraphs: chordality and conformality

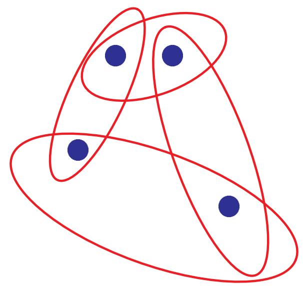
equivalent (for finite hypergraphs):

H fully reducible via deletions of

- $a \in A$ that are not multiply covered by S
- $s \in S$ if $s \subseteq s'$ for some $s' \in S$

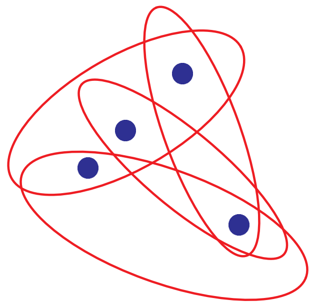
$G(H)$ has neither

irreducible cycles **chordal: no bad cycles**



nor

unguarded cliques **conformal: no bad cliques**



cycles and unguarded cliques
are broken up in unravellings

bisimilar covers: bisimilar companions "above"

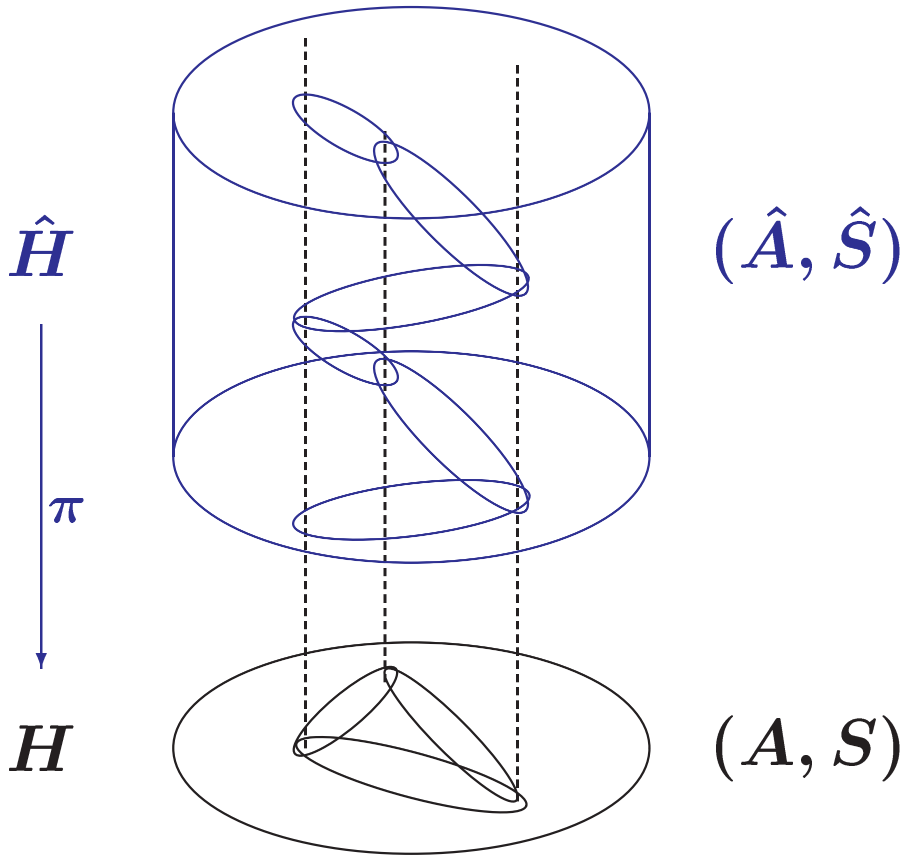
bisimilar hypergraph cover

of $H = (A, S)$ by $\hat{H} = (\hat{A}, \hat{S})$

$$\boldsymbol{\pi} : \quad \hat{H} \rightarrow H$$
$$\hat{H} \approx H$$

induced bisimulation

locally 1-1



unravellings provide

(infinite) **acyclic covers**

finite conformal covers

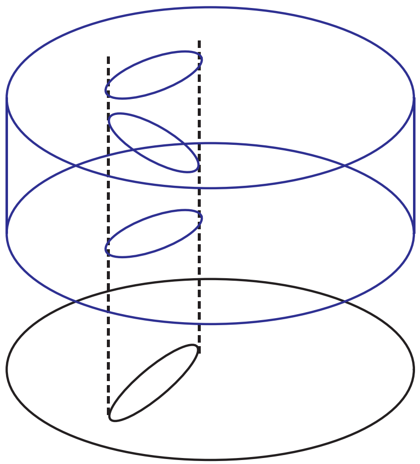
theorem (Hodkinson/O)

every finite hypergraph H admits a cover

$\boldsymbol{\pi \colon \hat{H} \xrightarrow{\approx} H}$ by a **finite conformal** $\boldsymbol{\hat{H}}$

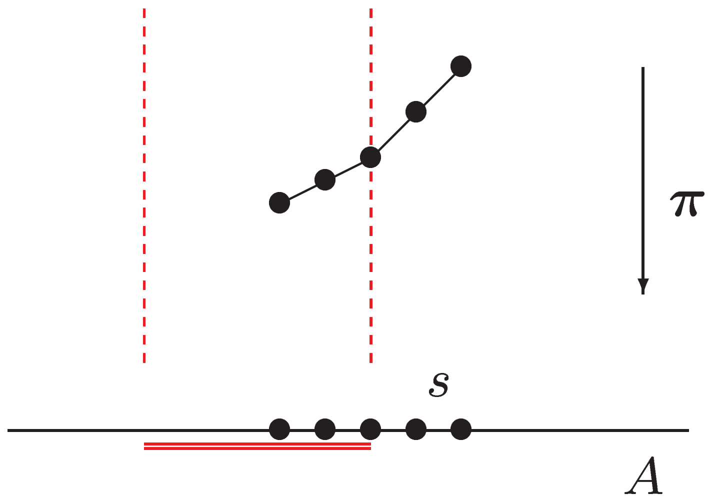
the construction is canonical and symmetry preserving:

- $\mathbf{Aut}(\boldsymbol{H}) \hookrightarrow \mathbf{Aut}(\boldsymbol{\hat{H}})$
- $\mathbf{Aut}_{\boldsymbol{\pi}}(\boldsymbol{\hat{H}})$ **transitive on** $\{\boldsymbol{\hat{s}} \in \boldsymbol{\hat{S}} \colon \boldsymbol{\pi}(\boldsymbol{\hat{s}}) = \boldsymbol{s}\} \subseteq \boldsymbol{\hat{S}}$



finite conformal covers: the construction

$$\pi\colon \hat{H} = (\hat{A}, \hat{S}) \xrightarrow{\approx} H = (A, S)$$



geometric intuition:
local "generic" cross-sections
in fibre bundle over base set

critical subsets: $U := \{ u \subseteq A \colon |u| > 1, s \in S \Rightarrow u \not\subseteq s \}$

$\uparrow$-closed set of unguarded/large sets
$u \in U$ must not be covered by clique

the universe

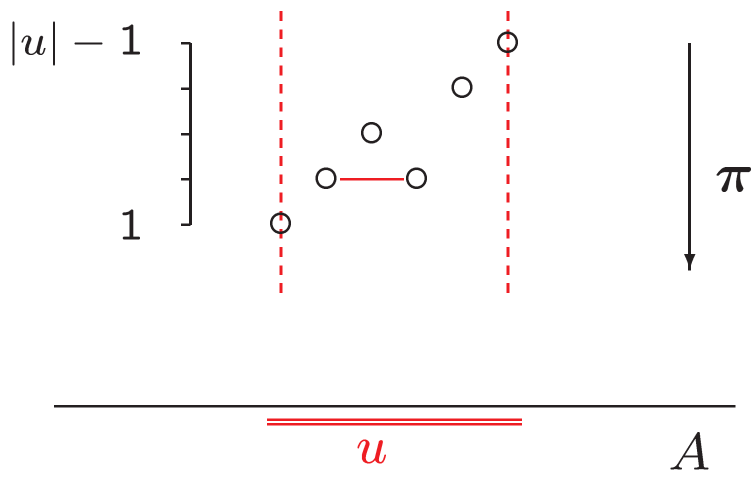
$$\hat{\mathbf{A}}\colon \begin{cases} (a, \chi_a) \in A \times {}^{U}\mathbb{N} \\ \chi_a(u) < |u| \,,\ \chi_a(u) \neq 0 \Leftrightarrow a \in u \end{cases}$$

generic subsets $\hat{s} \subseteq \hat{A}$

local cross sections over A: $a \mapsto (a, \chi_a)$ where

$\chi\_(u)\colon a \mapsto \chi_a(u)$ **one-to-one above every** u

for cardinality reasons: $\pi(\hat{s}) \notin U$

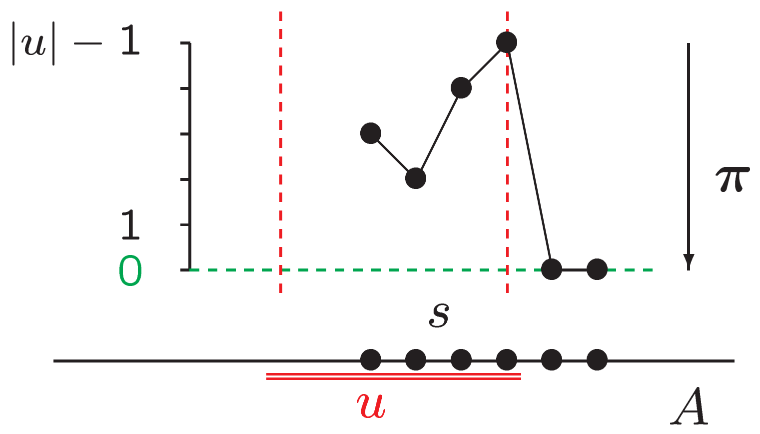


covering any $s \in S$ (the "back" property for $\approx$)

$$\hat{\mathrm{S}} := \{\hat{s} \subseteq \hat{A} \colon \hat{s} \text{ generic}, \pi(\hat{s}) \in S\}$$

- **any generic $t \subseteq \pi^{-1}[s]$ extends to $\hat{s} \in S$ above s**

  use $|s \cap u| < |u|$



conformality

no cliques above $u \in U$:

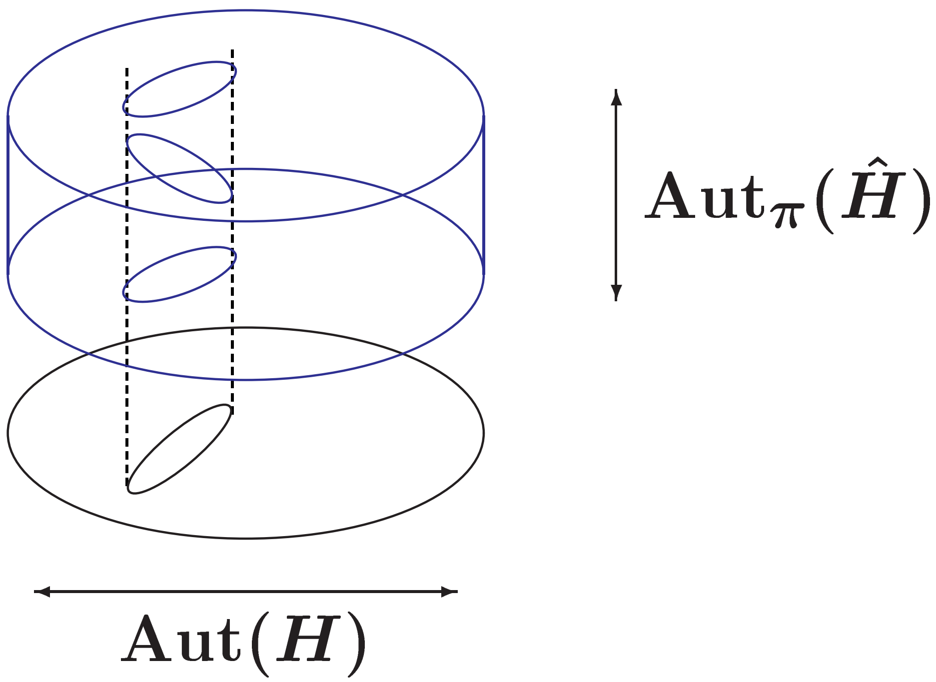
$\widehat{c}$ clique w.r.t. $\widehat{S}$
$\Rightarrow$ $\pi(\widehat{c}) \subseteq s$ for some $s \in S$
$\Rightarrow$ $\widehat{c} \subseteq \widehat{s} \in \widehat{S}$ guarded

automorphism properties of $\pi\colon \hat{H} \xrightarrow{\approx} H$

- $\mathbf{Aut}(H) \hookrightarrow \mathbf{Aut}(\hat{H})$ (by construction)

- $\mathbf{Aut}_{\pi}(\hat{H})$ **transitive on** $\{\hat{s} \in \hat{S} \colon \pi(\hat{s}) = s\}$

  (as in extensions for "back" property)



back to relational structures

let $H = (B, S) \succcurlyeq H(\mathfrak{B})$

all guarded subsets of $\mathfrak{B}$ covered by $s \in S$

$$\pi \colon \hat{H} \xrightarrow{\approx} H, \quad \hat{H} = (\hat{B}, \hat{S})$$

canonical $\hat{\mathfrak{B}}$ on $\hat{B}$ with $\hat{H} \succcurlyeq H(\hat{\mathfrak{B}})$

$$\boxed{\begin{array}{lll} \hat{\mathfrak{B}} := \big(\hat{B}, R^{\hat{\mathfrak{B}}}, \ldots\big) & & \\ \hat{\boldsymbol{b}} \in R^{\hat{\mathfrak{B}}} & \Leftrightarrow & [\hat{\boldsymbol{b}}] \subseteq \hat{s} \text{ for some } \hat{s} \in \hat{S} \\ & & \text{and } \pi(\hat{\boldsymbol{b}}) \in R^{\mathfrak{B}} \end{array}}$$

automorphism properties:

- $\mathrm{Aut}_S(\mathfrak{B}) \hookrightarrow \mathrm{Aut}(\hat{\mathfrak{B}})$
- $\pi \restriction \hat{s} \in \mathrm{Part}(\hat{\mathfrak{B}}, \mathfrak{B})$

  $\mathfrak{B} \restriction s$ embeds isomorphically into any $\hat{s} \in \hat{S}$ above $s \in S$
- $\mathrm{Aut}_\pi(\hat{\mathfrak{B}})$ **transitive on these embeddings**

finite conformal covers: their use for EPPA

from ordinary to tight EPPA extensions

from (ordinary) EPPA extension $\mathfrak{A} \subseteq \mathfrak{B}$ get

$$\boxed{\begin{array}{l} H := (B, S) \text{ where } \boldsymbol{S} = \{\boldsymbol{f}[\boldsymbol{A}] \colon \boldsymbol{f} \in \mathbf{Aut}(\mathfrak{B})\} \\ \textbf{conformal cover } \boldsymbol{\pi} \colon \hat{\boldsymbol{H}} \xrightarrow{\approx} \boldsymbol{H} \\ \qquad \text{with induced } \boldsymbol{\pi} \colon \hat{\mathfrak{B}} \longrightarrow \mathfrak{B} \end{array}}$$

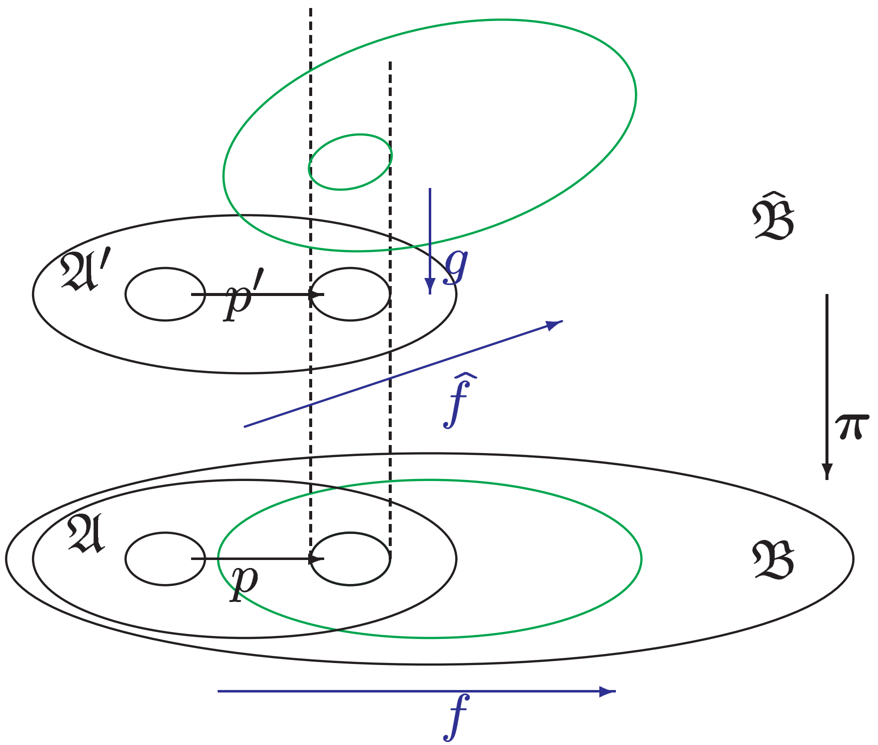
- $\mathfrak{A} = \mathfrak{B} \restriction A$ embeds as $\mathfrak{A} \simeq \mathfrak{A}' \subseteq \hat{\mathfrak{B}}$
- $\mathfrak{A}' \subseteq \hat{\mathfrak{B}}$ **EPPA and tight w.r.t. Gaifman cliques**
- $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}_S\mathfrak{B} \hookrightarrow \mathrm{Aut}(\hat{\mathfrak{B}})$

$\mathfrak{A}' \subseteq \hat{\mathfrak{B}}$ is an EPPA extension, tight w.r.t. Gaifman cliques

$p \in \text{Part}(\mathfrak{A})$ induces $p' \in \text{Part}(\mathfrak{A}')$

$\boldsymbol{f \in \mathbf{Aut}(\mathfrak{B})}$: $\boldsymbol{f \supseteq p}$ **lifts to** $\boldsymbol{\hat{f} \in \mathbf{Aut}(\hat{\mathfrak{B}})}$

but in general $\hat{f} \not\supseteq p'$



$$\boldsymbol{\hat{f}' := g \circ \hat{f} \supseteq p'}$$

clique tight extension thm

EPPA extensions without “new” Gaifman cliques

corollaries

EPPA for

- K_n**-free graphs** Herwig/Lascar
- **Henson digraphs** Herwig/Lascar
- **conformal relational structures**

guarded logics

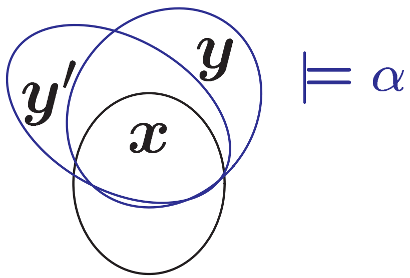
Andréka, van Benthem, Németi

quantification relativised
to relational (hyper-)edges

$$\forall \boldsymbol{y}\Big(\alpha(\boldsymbol{x},\boldsymbol{y}) \ \rightarrow \varphi(\boldsymbol{x},\boldsymbol{y})\Big)$$

guard atom: free(φ) $\subseteq$ var(α)

guarded subsets $[\boldsymbol{a}]$ for $\boldsymbol{a} \in R^{\mathfrak{A}}$
as "visible patches"



Guarded Fragment GF $\subseteq$ FO

subsumes ML (and extensions: $=$, universal/inverse modalities)

mirrors modal model theory in richer setting:
decidable, models of bounded tree width, finite models, . . .

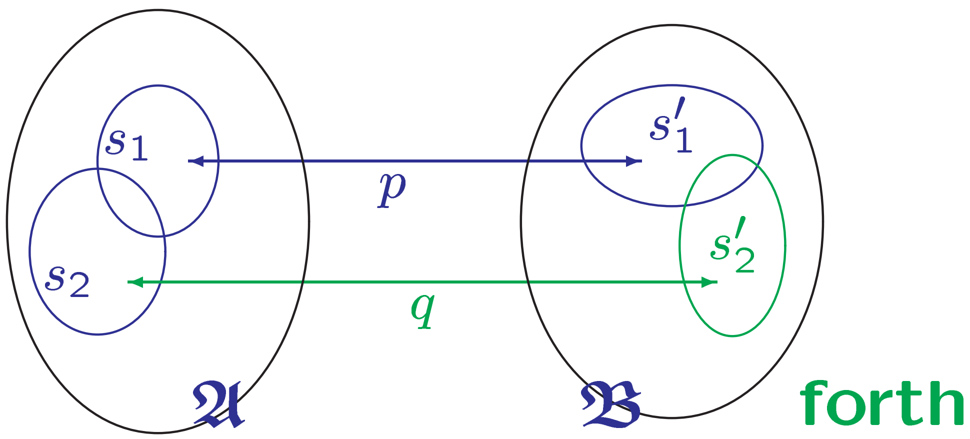
variation: clique guarded fragment CGF

guarded $\rightarrow$ clique guarded subsets as "visible patches"

guarded bisimulation – back & forth
Ehrenfeucht-Fraïssé for guarded quantification

$Z \colon \mathfrak{A} \sim_{\mathrm{g}} \mathfrak{B}$

$Z \colon H(\mathfrak{A}) \approx H(\mathfrak{B})$ with $Z \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$



$\sim_{\mathrm{g}}$ invariance of GF: **models of bounded tree width**
via unravelling

reduction to model theory of trees

finite conformal guarded covers

guarded covers: covers w.r.t. $\sim_{\mathrm{g}}$ rather than just $\approx$

theorem (Hodkinson/O)

for every finite relational $\mathfrak{A}$

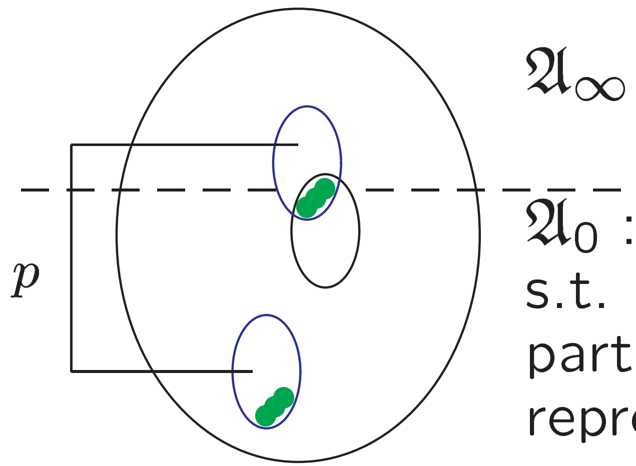
there is a **guarded cover** $\pi \colon \hat{\mathfrak{A}} \xrightarrow{\sim_{\mathrm{g}}} \mathfrak{A}$

by a **finite conformal** $\hat{\mathfrak{A}}$

with further application to finite model theory of guarded logics

EPPA and the finite model property for GF, CGF

finite closure of partial model (after Skolemization)



$\mathfrak{A}_\infty \models \varphi$

$\mathfrak{A}_0$: a large finite segment s.t. all missing witnesses have partially isomorphic representatives in $\mathfrak{A}_0$

φ holds in EPPA extensions of $\mathfrak{A}_0$ that are tight w.r.t.

- guarded sets [for GF] Grädel
- clique guarded sets [for CGF] Hodkinson, O

Open

can short bad cycles also be avoided in bisimilar covers?

only known for vocabulary of width 2 (graph-like)
here semi-direct products with
Cayley graphs of large girth work [O 02]

counterpart for Gaifman tight EPPA
in topolgy of the free group?

can other classes of configuartions be avoided in EPPA?