Amalgamation, Groupoids, Symmetries in Finite Structures

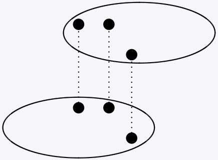
Martin Otto
TU Darmstadt

Leeds 11/2015

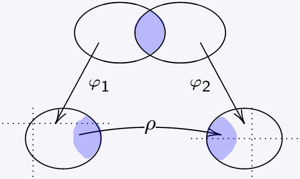
Global Finite Realisations of Local Specifications

examples of local views & specifications

- **exploded view of a hypergraph**



- **coordinate charts for manifolds**



- **decomposition and synthesis** of graphs, hypergraphs, ...
- **implicit specifications** of (macro-)bisimulation types $p \to \Diamond q$
- **i.e., guarded extension properties** $\forall \mathbf{x}\big(\theta(\mathbf{x}) \to \exists \mathbf{y}\theta'(\mathbf{xy})\big)$

local ——— global

atlas of local maps with changes of coordinates	**manifolds**
distinguished substructures with overlap specifications	**relational structures**
hyperedges with overlap specifications	**hypergraphs**
partial isomorphisms with composition in overlaps	**automorphism groups**

local specifications? **global realisations?**

the role of groupoids/inverse semigroups

two 'equivalent' algebraic formats for
composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

the role of groupoids/inverse semigroups

two 'equivalent' algebraic formats for
composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

groups capture global symmetries

groupoids capture local/partial symmetries

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

intersection graph of $\mathcal{A}$:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') \colon s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges $\mathbf{s} \in \mathbf{S}$

the role of hypergraphs

hypergraph: $\mathcal{A} = (A, S)$ with sets $\begin{cases} A \text{ of vertices} \\ S \subseteq \mathcal{P}(A) \text{ of hyperedges} \end{cases}$

examples:

- **combinatorial patterns of structural decompositions**
- **hypergraphs of guarded subsets of relational structures**

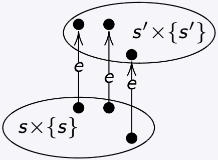
intersection graph of $\mathcal{A}$:

$\mathrm{I}(\mathcal{A}) := (S, E)$ where $E = \{(s, s') \colon s \neq s', s \cap s' \neq \emptyset\}$

records pairwise overlaps between hyperedges s $\in$ S

exploded view of $\mathcal{A}$ based on $\mathrm{I}(\mathcal{A})$

the disjoint union of the hyperedges $s \in S$
with partial bijections ρ_e for $e = (s, s') \in E$



$\rightsquigarrow$ **format of local overlap specifications**

overview

(I) specification & realisation of overlap patterns

(II) reduced products with groupoids (core results)

(III) from local to global symmetries

(IV) further applications

(I) abstract specification & realisation

incidence pattern $\mathrm{I} = (S, (E[s,s'])_{s,s' \in S})$

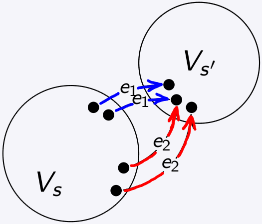
multi-graph with vertices $s \in S$ (sorts)
directed edges $e \in E[s,s']$ from s to s' with $e^{-1} \in E[s',s]$

- **fixed bisimulation type for pairwise overlaps**

I-**graph** $\mathrm{H} = \big(V, (V_s)_{s \in S}, (\rho_e)_{e \in E}\big)$

vertex set V partitioned into sorts V_s for $s \in S$
ρ_e a partial* bijection between V_s and $V_{s'}$ for $e \in E[s,s']$

- **an exploded view of the desired pairwise overlaps**



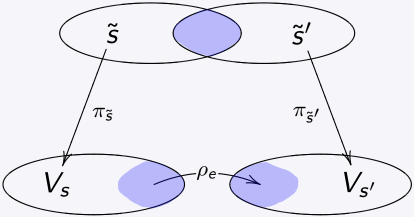
realisation:

a realisation of $\mathrm{H} = (V, (V_s), (\rho_e))$ is a
hypergraph $\mathcal{A} = (A, \tilde{S})$ with projection $\pi \colon \tilde{S} \longrightarrow S$
and an atlas of bijections $\pi_{\tilde{s}} \colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

realisation:

a realisation of $\mathrm{H} = (V, (V_s), (\rho_e))$ is a **hypergraph** $\mathcal{A} = (A, \tilde{S})$ with projection $\pi\colon \tilde{S} \longrightarrow S$ and an atlas of bijections $\pi_{\tilde{s}}\colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

- **all specified overlaps are realised:**
  for $e \in E[s, s']$, ρ_e is realised at every $\tilde{s} \in \pi^{-1}(s)$ by an actual overlap with some $\tilde{s}' \in \pi^{-1}(s')$

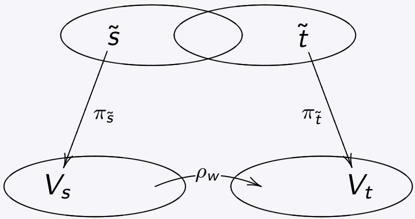


realisation:

a realisation of $\mathrm{H} = (V, (V_s), (\rho_e))$ is a
hypergraph $\mathcal{A} = (A, \tilde{S})$ with projection $\pi \colon \tilde{S} \longrightarrow S$
and an atlas of bijections $\pi_{\tilde{s}} \colon \tilde{s} \to V_{\pi(\tilde{s})}$ for $\tilde{s} \in \tilde{S}$ s.t.

- **all specified overlaps are realised:**
  for $e \in E[s, s']$, ρ_e is realised at every $\tilde{s} \in \pi^{-1}(s)$
  by an actual overlap with some $\tilde{s}' \in \pi^{-1}(s')$

- **no further, incidental overlaps occur:**
  all actual overlaps of $\mathrm{I}(\mathcal{A})$ are induced by
  compositions ρ_w of partial bijections ρ_e in H



realisations vs. exploded views

the exploded view of hypergraph $\mathcal{A} = (A, S)$
is an I-graph $\mathrm{H}(\mathcal{A})$ w.r.t. $\mathrm{I}(\mathcal{A}) = (S, E)$

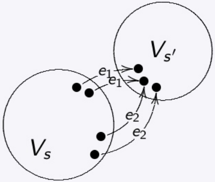
$\mathcal{A}$ is a realisation of $\mathbf{H}(\mathcal{A})$ obtained as a quotient $\mathrm{H}(\mathcal{A})/{\approx}$
w.r.t. $\approx$ induced by identifications encoded in the ρ_e of $\mathrm{H}(\mathcal{A})$

realisations vs. exploded views

the exploded view of hypergraph $\mathcal{A} = (A, S)$
is an I-graph $\mathrm{H}(\mathcal{A})$ w.r.t. $\mathrm{I}(\mathcal{A}) = (S, E)$

$\mathcal{A}$ is a realisation of $\mathrm{H}(\mathcal{A})$ obtained as a quotient $\mathrm{H}(\mathcal{A})/{\approx}$
w.r.t. $\approx$ induced by identifications encoded in the ρ_e of $\mathrm{H}(\mathcal{A})$

in general $\mathrm{H}/{\approx}$ may fail to realise H:
$\approx$ may even collapse individual V_s

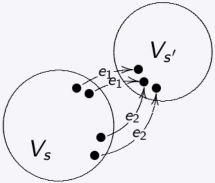


realisations vs. exploded views

the exploded view of hypergraph $\mathcal{A} = (A, S)$
is an I-graph $\mathrm{H}(\mathcal{A})$ w.r.t. $\mathrm{I}(\mathcal{A}) = (S, E)$

$\mathcal{A}$ is a realisation of $\mathrm{H}(\mathcal{A})$ obtained as a quotient $\mathrm{H}(\mathcal{A})/\!\approx$
w.r.t. $\approx$ induced by identifications encoded in the ρ_e of $\mathrm{H}(\mathcal{A})$

in general $\mathrm{H}/\!\approx$ may fail to realise H:
$\approx$ may even collapse individual V_s



idea: try local unfolding in products of H with . . . ?

(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

- **I-groupoids come with Cayley graphs that are I-graphs**

(II) reduced products with groupoids

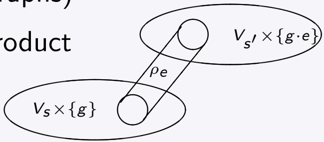
I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with
associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

- **I-groupoids come with Cayley graphs that are I-graphs**

reduced products as candidate realisations:

$\rightsquigarrow$ $\mathrm{H} \times \mathbb{G}$ natural direct product (of I-graphs)

$\rightsquigarrow$ $\mathrm{H} \otimes \mathbb{G} := (\mathrm{H} \times \mathbb{G})/\!\approx$ reduced product



(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

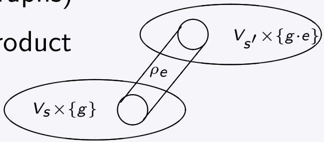
associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

- I-groupoids come with Cayley graphs that are I-graphs

reduced products as candidate realisations:

$\rightsquigarrow$ $\mathrm{H} \times \mathbb{G}$ natural direct product (of I-graphs)

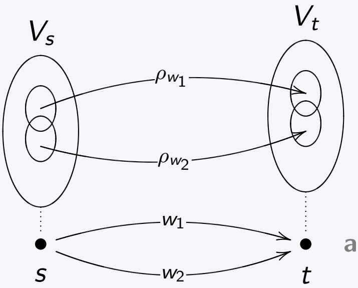
$\rightsquigarrow$ $\mathrm{H} \otimes \mathbb{G} := (\mathrm{H} \times \mathbb{G})/\!\approx$ reduced product



when is this a realisation of H?

obstructions to simple realisations

- H may fail to be *coherent* (as seen before): a lack of path-independence in H, with conflicting identifications collapsing individual V_s

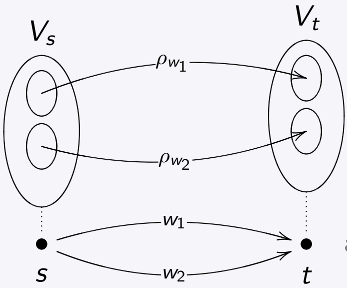


a bad cycle in $\mathbb{G}$?

can be overcome by relatively simple pre-processing

obstructions to simple realisations

- $\mathbb{H}$ and $\mathbb{G}$ may fail to be *confluent* in the product: causing incidental overlaps (with potential conflicts at the relational level)

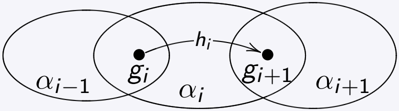


a bad cycle in $\mathbb{G}$?

$\rightsquigarrow$ **need substantial acyclicity conditions on $\mathbb{G}$**

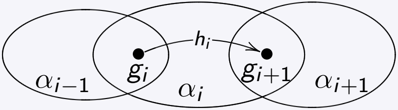
an appropriate notion of acyclicity

- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$

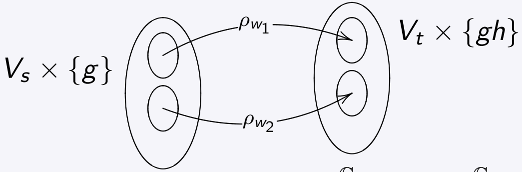


an appropriate notion of acyclicity

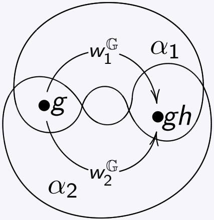
- not just short cycles in the Cayley graph of $\mathbb{G}$, but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



- in particular, need to avoid certain coset cycles of length 2



$$h_1 = w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}} = h_2^{-1}$$



any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

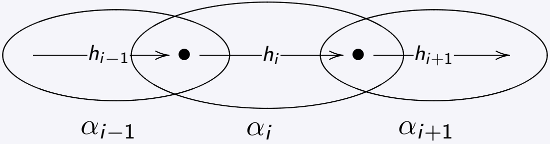
any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

construction by inductive interleaving:

- groupoidal action on I-graphs
- use of amalgamation chains of $\mathrm{I}[\alpha]$-graphs (local unfoldings) to eliminate short cycles



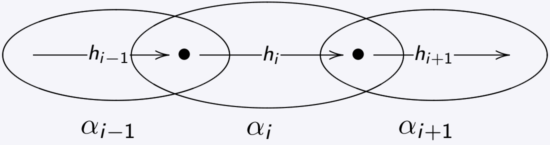
any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

construction by inductive interleaving:

- groupoidal action on I-graphs
- use of amalgamation chains of $\mathrm{I}[\alpha]$-graphs (local unfoldings) to eliminate short cycles



cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H}$ (an I-graph), obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H}$ (an I-graph), obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

symmetric realisations

any degree of acyclicity in symmetric realisations

theorem (O_13)

for any overlap specification $\mathbb{H}$ (an I-graph), obtain realisations $\mathbb{H} \otimes \mathbb{G}$ (as reduced products with finite I-groupoids $\mathbb{G}$) that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\mathbb{H}$

symmetric realisations

corollary

every finite hypergraph admits, for $\mathsf{N} \in \mathbb{N}$, finite coverings that

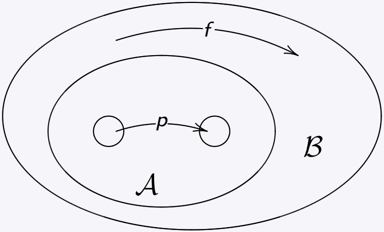
- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

(III) from local to global symmetries

extension property for partial automorphisms (EPPA):
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$



(III) from local to global symmetries

extension property for partial automorphisms (EPPA):
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

---

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$
s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$

theorem (Herwig–Lascar 00)

---

same, as a *finite model property* over any class $\mathcal{C}$
defined by finitely many forbidden homomorphisms

if $\mathcal{A} \in \mathcal{C}_{\mathrm{fin}}$ has any EPPA extension in $\mathcal{C}$
then it also has a finite one in $\mathcal{C}_{\mathrm{fin}}$

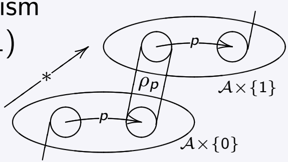
a "naive" idea towards EPPA for single p

for a single partial isomorphism p of $\mathcal{A} = (A, R)$ find

a free infinite EPPA extension as a reduced product $(\mathcal{A} \times \mathbb{Z})/{\approx}$
where $\approx$ is induced by the partial bijections

$$\begin{array}{rcl} \rho_p^{i,i-1} \colon \mathcal{A} \times \{i\} & \longrightarrow & \mathcal{A} \times \{i-1\} \\ (a,i) & \longmapsto & (p(a), i-1) \end{array}$$

p in $\mathcal{A} \simeq \mathcal{A} \times \{0\}$ extends to the automorphism
induced by the shift $* \colon (a,i) \longmapsto (a, i+1)$



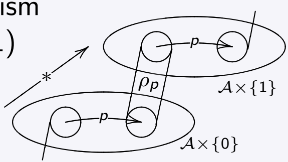
a "naive" idea towards EPPA for single p

for a single partial isomorphism p of $\mathcal{A} = (A, R)$ find

a free infinite EPPA extension as a reduced product $(\mathcal{A} \times \mathbb{Z})/\!\approx$ where $\approx$ is induced by the partial bijections

$$\begin{array}{rccc} \rho_p^{i,i-1} \colon \mathcal{A} \times \{i\} & \longrightarrow & \mathcal{A} \times \{i-1\} \\ (a,i) & \longmapsto & (p(a), i-1) \end{array}$$

p in $\mathcal{A} \simeq \mathcal{A} \times \{0\}$ extends to the automorphism induced by the shift $\ast \colon (a,i) \longmapsto (a, i+1)$



and from that a finite EPPA extension as a quotient $(\mathcal{A} \times \mathbb{Z}_n)/\!\approx$ for any $n \geqslant 3$ such that $p^n = \mathrm{id}_s$

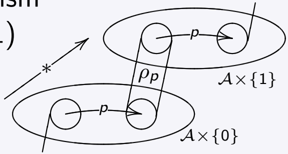
a “naive” idea towards EPPA for single p

for a single partial isomorphism p of $\mathcal{A} = (A, R)$ find

a free infinite EPPA extension as a reduced product $(\mathcal{A} \times \mathbb{Z})/\!\approx$ where $\approx$ is induced by the partial bijections

$$\begin{array}{rcl} \rho_p^{i,i-1} \colon \mathcal{A} \times \{i\} & \longrightarrow & \mathcal{A} \times \{i-1\} \\ (a, i) & \longmapsto & (p(a), i-1) \end{array}$$

p in $\mathcal{A} \simeq \mathcal{A} \times \{0\}$ extends to the automorphism induced by the shift $* \colon (a, i) \longmapsto (a, i+1)$



and from that a finite EPPA extension as a quotient $(\mathcal{A} \times \mathbb{Z}_n)/\!\approx$ for any $n \geqslant 3$ such that $p^n = \mathrm{id}_s$

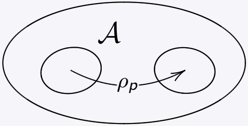
groupoidal realisations can do the trick for several p (!)

new proof of full Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification
for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I(\mathcal{A}, P)}$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H(\mathcal{A}, P)}$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!

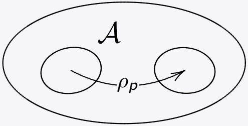


new proof of full Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification
for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I}(\mathcal{A}, \mathbf{P})$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H}(\mathcal{A}, \mathbf{P})$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!



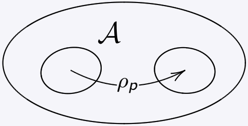
(iii) **symmetric realisations of $\mathbf{H}(\mathcal{A}, \mathbf{P})$ are EPPA extensions !**

new proof of full Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I}(\mathcal{A}, \mathbf{P})$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\mathbf{H}(\mathcal{A}, \mathbf{P})$:**
$\mathrm{I}(\mathcal{A}, P)$-graph $\mathrm{H}(\mathcal{A}, P) = (A, (\rho_p)_{p \in P})$
needs to be made coherent!



(iii) **symmetric realisations of $\mathbf{H}(\mathcal{A}, \mathbf{P})$ are EPPA extensions !**

(iv) **N-acyclic EPPA extensions are N-free:**
admit N-local homomorphisms into every (finite or infinite) EPPA extension due to their N-local tree-decomposability

(IV) applications in algorithmic model theory

- **characterisation theorems** (fmt)
  for guarded logics and relatives

  using finite coverings of controlled acyclicity

- **finite model properties & finite controllability**
  for guarded logics and constraints

  using finite coverings of controlled acyclicity
  and/or Herwig–Lascar extension properties

⟶ **O**_(LICS10&JACM13)
O_(APAL13)
Bárány–Gottlob–O_(LICS10&LMCS14)
Bárány–ten Cate–O_(VLDB12)
O_(LICS13&arXiv14/15)

characterisation theorems (fmt)

theorem (O_10)

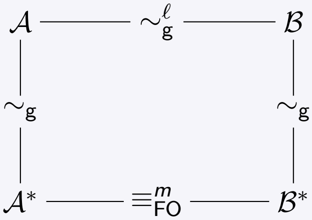
$\mathsf{GF} \equiv \mathsf{FO}/\!\sim_{\mathsf{g}}$ and $\mathsf{GF} \equiv_{\mathsf{fin}} \mathsf{FO}/\!\sim_{\mathsf{g}}$

characterisation theorems (fmt)

theorem (O_10)

$\mathrm{GF} \equiv \mathrm{FO}/{\sim_{\mathrm{g}}}$ and $\mathrm{GF} \equiv_{\mathrm{fin}} \mathrm{FO}/{\sim_{\mathrm{g}}}$

idea: show that $\sim_{\mathrm{g}}$-invariance of $\varphi \in \mathrm{FO}^m$ implies $\sim_{\mathrm{g}}^{\ell}$-invariance for some $\ell = \ell(m)$ such that over suitable locally sufficiently acyclic (finite) structures, $\sim_{\mathrm{g}}^{\ell}$ refines $\equiv_{\mathrm{FO}}^{m}$



finite model properties & finite controllability

theorem (Bárány–Gottlob–O_10/Rosati06)

finite controllability for union of conjunctive queries Q
w.r.t. constraint $\alpha \in \mathsf{GF}$:

$$\alpha \models Q \quad \Leftrightarrow \quad \alpha \models_{\mathsf{fin}} Q \qquad (\dagger)$$

finite model properties & finite controllability

theorem (Bárány–Gottlob–O_10/Rosati06)

finite controllability for union of conjunctive queries Q
w.r.t. constraint $\alpha \in \mathrm{GF}$:

$$\alpha \models Q \quad \Leftrightarrow \quad \alpha \models_{\mathrm{fin}} Q \qquad (\dagger)$$

idea 1: detour via "treeification" $Q^* \in \mathrm{GF}$ of Q for which

$$Q^* \models Q \text{ and } Q \models_{\mathrm{acyc}} Q^*;$$

use locally sufficiently acyclic (finite) unfoldings to show that also

$$\alpha \models_{\mathrm{fin}} Q \;\Rightarrow\; \alpha \models_{\mathrm{fin}} Q^*.$$

finite model properties & finite controllability

theorem (Bárány–Gottlob–O_10/Rosati06)

finite controllability for union of conjunctive queries Q
w.r.t. constraint $\alpha \in \mathrm{GF}$:

$$\alpha \models Q \quad \Leftrightarrow \quad \alpha \models_{\mathrm{fin}} Q \qquad (\dagger)$$

idea 1: detour via "treeification" $Q^* \in \mathrm{GF}$ of Q for which

$$Q^* \models Q \text{ and } Q \models_{\mathrm{acyc}} Q^*;$$

use locally sufficiently acyclic (finite) unfoldings to show that also

$$\alpha \models_{\mathrm{fin}} Q \;\Rightarrow\; \alpha \models_{\mathrm{fin}} Q^*.$$

idea 2: view ($\dagger$) as a fmp for $\alpha \in \mathrm{GF}$ within $\mathcal{C} = \mathrm{Mod}(\neg Q)$,
which follows from Herwig–Lascar EPPA (!)

summary

- **a generic construction of**
  highly acyclic finite groupoids
- **a universal & generic route to the synthesis of**
  finite realisations (and coverings) in reduced products
- **symmetry and acyclicity of realisations supports**
  extensions of local to global symmetry
- **further applications in finite model theory**

$\longrightarrow$ **Finite Groupoids, Finite Coverings**
& Symmetries in Finite Structures (arXiv 2015 (v4))