Finite Model Constructions for Guarded Logics

FMT in Les Houches 2012 Martin Otto

- guardedness and guarded logics
- hypergraph/guarded bisimulation
- finite model properties
- expressive completeness

the hypergraph of guarded subsets

of a relational structure $\mathcal{A} = (A, (R^{\mathcal{A}})_{R \in \tau})$:

$$\mathbf{H}(\mathcal{A}) = (\mathbf{A}, \mathbf{S}[\mathcal{A}])$$

with hyperedges generated by subsets $[\mathbf{a}] \subseteq A$ for $\mathbf{a} \in R^{\mathcal{A}}$, $R \in \tau$
closed under subsets & singleton sets

hypergraph terminology:

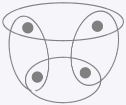
- $\mathbf{H} = (\mathbf{A}, \mathbf{S})$, $S \subseteq \mathcal{P}(A)$ the set of hyperedges
- $\mathbf{G}(\mathbf{H}) = (\mathbf{A}, \mathbf{E})$, associated graph: hyperedges $\rightsquigarrow$ cliques
- $\mathbf{G}(\mathcal{A}) = \mathbf{G}(\mathbf{H}(\mathcal{A}))$, the Gaifman graph of $\mathcal{A}$

guarded sets & link structure

relational content vs. **hypergraph link structure** (topology)

guarded links: local overlaps,
hyperedge incidence
local & global aspects

tuples a ∈ R



different mixes for different purposes, but

- hypergraph 'topology' matters
- tree-likeness is good, locally or globally, if available

e.g., in relation to databases or CSP

active domain, conjunctive queries, tgd, ...

On the desirability of acyclic database schemas

Beeri–Fagin–Maier–Yannakakis 1983

attention to link structure (topology)

discrete mathematics: analogies between hypergraphs and graphs

logic, model theory: analogies between guarded and modal logics

databases: analogies between databases and transition systems

e.g., tree-decompositions play on such analogies

key question: how far do these analogies carry?

- global acyclicity not usually available in finite unfoldings
- combinatorics and model theory of hypergraphs

hypergraph acyclicity = tree decomposability

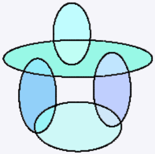
equivalent characterisations:

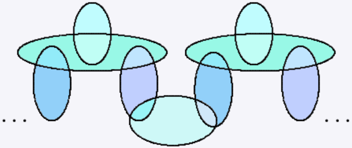
- $\mathbf{H} = (\mathbf{A}, \mathbf{S})$ admits reduction $\mathbf{H} \rightsquigarrow \emptyset$

  through deletion of $\begin{cases} a & \text{if } |\{s \colon a \in s\}| \leqslant 1 \\ s & \text{if } s \subsetneq s' \in S \end{cases}$

- $\mathbf{H}$ has tree decomposition $\delta \colon \mathcal{T} \to S$ with bag set S

- **H is conformal & chordal (later)**





NB: hypergraph tree-decompositions of $H(\mathcal{A})$ induce special tree-decompositions of $\mathcal{A}$

guarded logics

guarded subsets/tuples as basic observables

access to links and relational content in various protocols

with different levels of expressive power, e.g.

FO existential & tree-like	$\rightsquigarrow$	acyclic conjunctive queries
general FO existential	$\rightsquigarrow$	conjunctive queries CQ
FO with alternation, link-based	$\rightsquigarrow$	guarded fragment GF
fixpoints	$\rightsquigarrow$	guarded fixpoint logic μGF
second-order	$\rightsquigarrow$	guarded second-order GSO

guiding idea (A): relational analogues of modal logics

the guarded fragment GF:

atomic formulae of FO, booleans, and

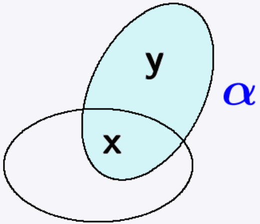
guarded quantification

$$\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})\big)$$

$$\forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big)$$

with guard atom α s.t.

$$\mathrm{free}(\varphi) \subseteq \mathrm{free}(\alpha) = \mathrm{var}(\alpha)$$



example: $\forall \mathbf{x}(R\mathbf{x} \rightarrow \exists \mathbf{y}(W\mathbf{xy} \wedge \neg Q\mathbf{y})) \quad \sim \quad \Box_\forall(r \rightarrow \Diamond_W \neg q)$

ML $\subsetneq$ **GF** $\subsetneq$ **FO**

guiding idea (B): guard against negation

Barany, ten Cate, Segoufin 2011

the guarded negation fragment GNF:

existential positive FO augmented by

guarded negation $\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \neg\varphi(\mathbf{x})\big)$
$=: \mathrm{gdd}(\mathbf{x}) \wedge \neg\varphi(\mathbf{x})$

GF $\subseteq_{\mathbf{g}}$ GNF (for guarded free variables!):

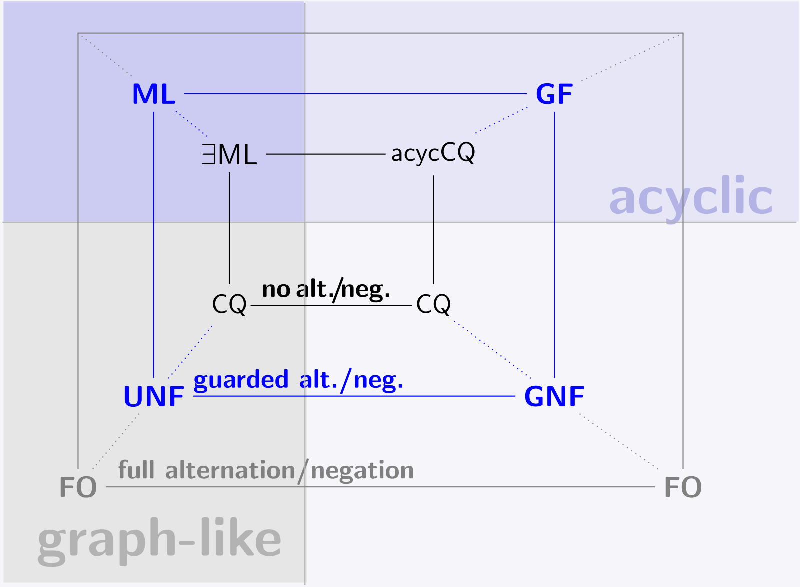
$$\mathrm{gdd}(\mathbf{x}) \wedge \forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big) \equiv \mathrm{gdd}(\mathbf{x}) \wedge \neg\, \exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \neg\varphi(\mathbf{x})\big)$$

- **GF $\equiv$ acycGNF** (with acyclic templates)
- **GNF $\supseteq$ UNF** generalises unary negation (ten Cate, Segoufin)

alternative picture (C): variation along three axes

graphs	$\longrightarrow$	**hypergraphs**
transition systems		general relational structures
ML		**GF**
UNF		**GNF**
existential positive	$\longrightarrow$	**∃/∀ alternation** (guarded or full)
		based on guarded or full negation
∃posFO		**GNF or FO**
acyclic templates	$\longrightarrow$	**general templates**
acycCQ		**CQ**
GF		**GNF**

where GF and GNF fit



bisimulation – the quintessential back&forth

- modal (two-way, global) bisimulation / graph bisimulation
- guarded bisimulation / hypergraph bisimulation
- guarded negation bisimulation / homomorphism bisimulation
  protocol mixing local homomorphisms with bisimulation

graph and hypergraph bisimulation:

local matches between local states (nodes or hyperedges)

maintained by player **II** against

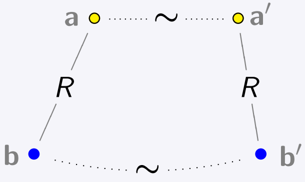
challenge/response w.r.t. links (edges or overlaps)

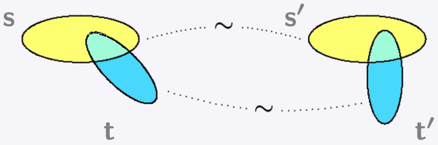
homomorphism bisimulation:

challenge/response w.r.t. (size-bounded) homomorphisms

graph and hypergraph bisimulation ∼

challenge/response — in graph and hypergraph bisimulation





● ⇝ ●, respecting R

● ⇝ ●, respecting partial overlap

with relational content:
replace local bijections by local isomorphisms
to obtain modal and guarded bisimulation

mutatis mutandis: clique guarded bisimulation for CGF

homomorphism bisimulation $\sim_{\text{hom}}$

local matches: bijections $\mathsf{s} \leftrightarrow \mathsf{s}'$ between guarded subsets

challenge/response:

- player **I** proposes subset $p \subseteq A$ (or $p' \subseteq A'$)
- player **II** chooses homomorphism $h \colon p \to A'$ compatible with $s \leftrightarrow s'$

h needs to be bijective on guarded subsets:

- player **I** chooses guarded $t \subseteq \mathrm{dom}(h)$
  new $t \leftrightarrow t'$: the restriction of h to t

k-size-bounded variants ($|h| \leqslant k$)**:** $(\mathbf{A}, \mathbf{S}) \sim_{\text{hom[k]}} (\mathbf{A}', \mathbf{S}')$

with relational content: $\mathcal{A} \sim_{\text{gn[k]}} \mathcal{A}'$
guarded negation bisimulation

Ehrenfeucht–Fraïssé and classical characterisations

bismulation invariance: $\sim_{\mathsf{g}}$ preserves GF
$\sim_{\mathsf{gn[k]}}$ preserves GNF[k]

Ehrenfeucht–Fraïssé correspondences, as expected

$\sim^{m}_{\mathsf{g}}$ — $\equiv^{m}_{\mathsf{GF}}$

$\sim^{m}_{\mathsf{gn[k]}}$ — $\equiv^{m}_{\mathsf{GNF[k]}}$

for relational structures with guarded tuples

characterisations of expressiveness, classically as expected

$\mathsf{FO}/\sim_{\mathsf{g}} \quad \equiv_{\mathsf{g}} \quad \mathsf{GF}$

$\mathsf{FO}/\sim_{\mathsf{gn[k]}} \quad \equiv_{\mathsf{g}} \quad \mathsf{GNF[k]}$

over relational structures with guarded tuples

bisimulation invariance

generalised tree model properties (Grädel 1999 for GF)

tree-like unfoldings yield special (infinite) models
of bounded treewidth for guarded logics like GF, GNF[k]

of importance for:

$\longrightarrow$ automata and model-checking games on trees

$\longrightarrow$ via interpretations, reductions to MSO on trees

$\longrightarrow$ characterisations of fixpoint extensions (classically)
via reduction to $\mathrm{MSO}/{\sim} \equiv \mathrm{L}_\mu$ (Janin–Walukiewicz 1996)

bisimilar covers – of graphs, hypergraphs & structures

graph and hypergraph covers:

$\pi \colon \hat{\mathbf{G}} \xrightarrow{\sim} \mathbf{G}$

graph homomorphism
inducing graph bisimulation
of local matches $(\hat{a}, \pi(\hat{a}))$

$\pi \colon \hat{\mathbf{H}} \xrightarrow{\sim} \mathbf{H}$

hypergraph homomorphism
inducing hypergraph bisimulation
of local bijections $\hat{s} \leftrightarrow \pi(\hat{s})$

homomorphisms with back-property w.r.t. link pattern

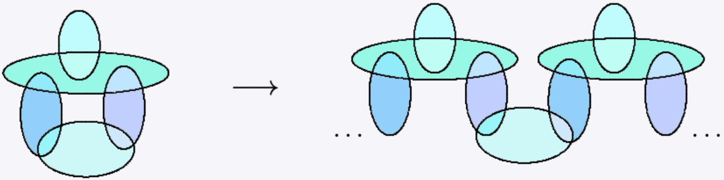
modal and guarded covers:

the same, with relational content

guarded covers are relational homomorphisms with back-property with induced hypergraph and guarded bisimulations

tree unfoldings as acyclic covers

observation: natural tree unfoldings w.r.t. link structure
of graphs, hypergraphs & structures
yield acyclic, albeit infinite, acyclic covers



fact: tree unfoldings of cyclic structures are infinite,
all acyclic covers of cyclic structures are infinite

how much acyclicity is possible in finite covers?

combinatorial challenge

(I) finite model constructions & fmp for guarded logics

- emphasis on hypergraph covers in model construction
- **relational Skolemisation + suitable covers = fmp**

examples:

(1) conformal covers
(2) covers with forbidden homomorphisms
(3) covers with forbidden cyclic configurations

reap fmp for CGF and GNF through Skolemisation
and the elimination of incidental links

first: finite hypergraphs and degrees of acyclicity

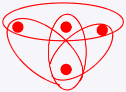
acyclicity = conformality + chordality

alternative characterisation of hypergraph acylicity:

conformality:

every clique in **G(H)** guarded

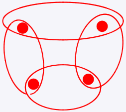
no



chordality:

every cycle of length $\geqslant 4$ in **G(H)** has a chord

no

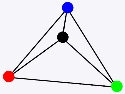


conformality can be achieved in finite covers, chordality cannot !

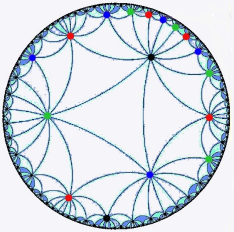
levels of acyclicity

conformality can be achieved in finite covers, chordality cannot !

even 1-local chordality may not be available in finite covers



locally finite cover of tetrahedron on ●, ●, ●, ●



relaxations:

N-chordality:
chordality for short cycles (of length $\leqslant$ N)

N-acyclicity:
acyclicity for small substructures (of size $\leqslant$ N)

conformal covers and fmp for CGF

conformal covers (Hodkinson–O 2003)

every finite hypergraph $\mathbf{H} = (\mathbf{A}, \mathbf{S})$
admits a finite conformal cover $\pi \colon (\hat{\mathbf{A}}, \hat{\mathbf{S}}) \xrightarrow{\sim} (\mathbf{A}, \mathbf{S})$

analogously, for finite relational structures $\mathcal{A}$:
finite conformal guarded covers $\pi \colon \hat{\mathcal{A}} \xrightarrow{\sim_g} \mathcal{A}$

application: reduction of FINSAT(**CGF**) **to** FINSAT(**GF**)

- expand by guards for required cliques
  force positive CGF-assertions
- eliminate incidental cliques in cover
  preserve negative CGF-assertions

covers and forbidden homomorphisms

weakly N-acyclic covers (Barany–Gottlob–O 2010)

every finite hypergraph $\mathbf{H} = (\mathbf{A}, \mathbf{S})$
admits finite weakly N-acyclic covers $\pi\colon (\hat{\mathbf{A}}, \hat{\mathbf{S}}) \xrightarrow{\sim} (\mathbf{A}, \mathbf{S})$

similarly for finite relational structures

weak N-acyclicity:
short cycles (length $\leqslant$ N) in cover may not be chordal
but acquire chords in projection s.t.

small homomorphic images in cover are acyclic in projection

application: small finite models of $\varphi \in \mathbf{GF}$
avoiding given homomorphisms/UCQ

covers and forbidden cyclic configurations

N-acyclic covers (O 2010)

every finite hypergraph $\mathbf{H} = (\mathbf{A}, \mathbf{S})$
admits finite N-acyclic covers $\pi \colon (\hat{\mathbf{A}}, \hat{\mathbf{S}}) \xrightarrow{\sim} (\mathbf{A}, \mathbf{S})$

similarly for finite relational structures

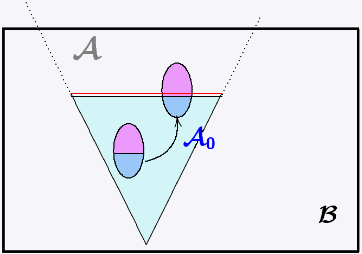
N-acyclicity:
all small induced sub-configurations are acyclic
$\longrightarrow$ **interesting structure theory**
based on local convex hulls

application: finite models of $\varphi \in \mathbf{GF}$
avoiding given set of cyclic substructures

fmp for GF

- Grädel 1999, based on Herwig's EPPA
  Hrushovski–Herwig–Lascar . . .

extension of partial isomorphisms to automorphisms
after relational Skolemisation ($\rightsquigarrow$ guarded $\forall\exists$)
from finite part of regular infinite model



- Barany–Gottlob–O 2010, based on weakly N-acyclic covers
  from pre-model: finite quotient of regular (infinite) model

summary of applications (so far)

positive link requirements covered by guarded $\forall\exists$ conditions
— after relational Skolemisation with extra guards

finite models for CGF

conformal covers break up false cliques

finite models for GNF and for GF avoiding UCQ

weakly N-acyclic covers break up false positives for CQ

small finite models and Ptime canonisation for GF

weakly N-acyclic covers break up relational inconsistencies

finite models for GF avoiding given cyclic configurations

N-acyclic covers break up false cyclic positives for CQ without ruling out acyclic positives !

(II) finite covers & expressive completeness/FO

expressive completeness thms: $\mathbf{FO}/{\sim_{\mathbf{L}}} \equiv \mathbf{L}$ **(fmt)**

where $\sim_{\mathsf{L}}$ is full L-bisimulation equivalence

partial L-isomorphy

$\sim^{\mathbf{m}}_{\mathbf{L}}$ corresponds to L^m-equivalence

L-Ehrenfeucht–Fraïssé

crux: a compactness property

if, for $\varphi \in \mathsf{FO}$ there is $m \in \mathbb{N}$ such that

φ $\sim_{\mathbf{L}}$**-invariant** $\Rightarrow$ φ $\sim^{\mathbf{m}}_{\mathbf{L}}$**-invariant,**

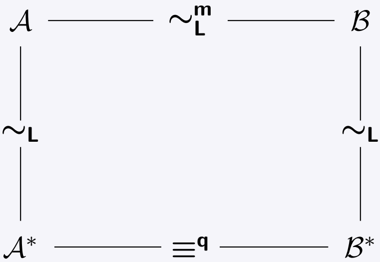
then $\mathsf{FO}/{\sim_{\mathsf{L}}} \equiv \mathsf{L}$ follows

focus here on GF and GNF

compactness property for expressive completeness

to show $\boxed{\varphi \sim_{\mathbf{L}}\text{-}\textbf{invariant} \;\Rightarrow\; \varphi \sim_{\mathbf{L}}^{\mathbf{m}}\text{-}\textbf{invariant}}$ for $\varphi \in \mathbf{FO}^q$

upgrade $\sim_{\mathbf{L}}^{\mathbf{m}}$ to $\equiv^{\mathbf{q}}$ in $\sim_{\mathsf{L}}$-equivalent finite companions:



$\mathcal{A} \mapsto \mathcal{A}^* \,/\, \mathcal{B} \mapsto \mathcal{B}^*$
special constructions
to break obstacles to $\equiv^{\mathbf{q}}$
while preserving $\sim_{\mathbf{L}}$-type

obtacles: small multiplicities, small cliques, and short cycles

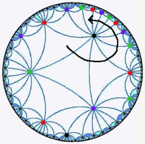
expressive completeness for GF

thm: $\mathrm{FO}/\!\sim_{\mathrm{g}} \;\equiv_{\mathrm{g}}\; \mathrm{GF}$ (fmt)

- upgrading uses **N-acyclic covers** and **finitary saturation**
  both based on highly acyclic Cayley groups
- over these richly branching covers: **E–F game analysis**
  based on structure theory of N-acyclic hypergraphs

two core ingredients:

Cayley groups that have no short cycles even
w.r.t. non-trivial transitions between cosets

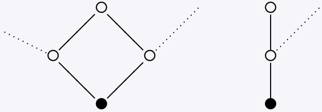


size-bounded local convex hulls of small configurations
so that GF determines isomorphism types, due to acyclicity

expressive completeness for GNF

thm: FO/$\sim_{\mathrm{gn[k]}}$ $\equiv_{\mathrm{g}}$ GNF[k] (fmt)

new obtacle for upgrading: non-isomorphic realisations of CQ



- local saturation w.r.t. distinct isomorphism types of small CQ
- relational Skolemisation to force positive CQ requirements
- N-acyclic covers to break false positives for CQ

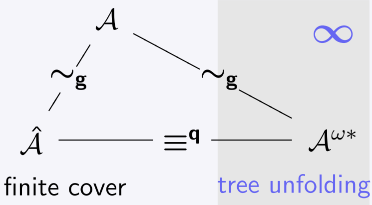
upgrading through $\sim^{\mathrm{m}}_{\mathrm{gn[k]}} \rightsquigarrow \sim^{\mathrm{m}}_{\mathrm{g}} \rightsquigarrow \equiv^{\mathrm{q}}$

intricate reduction to GF:
currently with detour through infinite tree-like models
+ fmp for GNF from Barany–ten Cate–Segoufin 2011

Skolemisation and upgrading for GNF

obeservation (for GF)

- finitely saturated, N-acyclic covers can serve as finite analogues of infinite ω-tree-unfoldings



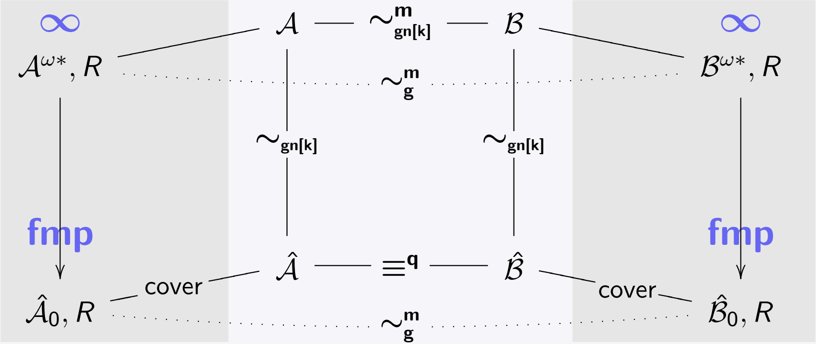
- use ω-unfoldings that branch on isomorphism types of small CQ ∞

- fmp for GNF (with Skolemisation for small CQ) generates finite pre-model, then N-acyclic covers . . .

expressive completeness for GNF

a glimpse of the complications:

core: upgrading in finite covers that behave like trees



summary & outlook

model-theoretic and algorithmic well-behavedness of guarded logics

nice model properties

expressive completeness through upgrading

approximations/counterparts of tree-like unfoldings

finite hypergraph constructions especially finite bisimilar covers

interesting problems, regarding
new methods, constructions, applications

$\longrightarrow$ work with Vince and Balder (GN DATALOG), ...
$\longrightarrow$ new dedicated project (DFG)

discrete mathematics – combinatorics – logic – and ...