Boundedness in Universal FO

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boundedness

least fixed point induction on $\varphi(\boldsymbol{X}, \boldsymbol{x})$, positive in X

$\boldsymbol{x}$ and X of matching arities r

least fixpoint $\boxed{(\mu_X \varphi)[\mathfrak{A}] = \bigcap \{ P \subseteq A^r \colon \varphi[\mathfrak{A}, P] \subseteq P \}}$

inductive stages: $(\mu_X \varphi)[\mathfrak{A}] = \bigcup_\alpha X^\alpha[\mathfrak{A}]$ where

$$\begin{aligned} X^0[\mathfrak{A}] &= \emptyset \\ X^{\alpha+1}[\mathfrak{A}] &= \varphi[\mathfrak{A}, X^\alpha[\mathfrak{A}]] \\ X^\lambda[\mathfrak{A}] &= \bigcup_{\alpha<\lambda} X^\alpha[\mathfrak{A}] \end{aligned}$$

closure ordinal $\gamma[\varphi, \mathfrak{A}] = \min_\alpha \big(X^{\alpha+1}[\mathfrak{A}] = X^\alpha[\mathfrak{A}] \big)$

$\varphi(\boldsymbol{X}, \boldsymbol{x})$ **bounded:** $\exists n \in \mathbb{N}$ **s.t.** $\gamma[\varphi, \mathfrak{A}] < n$ **for all** $\mathfrak{A}$

$\varphi(X, \boldsymbol{x}) \in$ FO bounded $\Rightarrow$ recursion spurious

$\Rightarrow$ $\mu_X \varphi$ uniformly FO

boundedness and definability

Barwise–Moschovakis theorem

for any FO formula $\varphi(X, x)$, postive in X,
the following are equivalent:

(i) $\mu_X\varphi$ **bounded**

(ii) $\mu_X\varphi$ **uniformly FO definable**

(iii) $\mu_X\varphi[\mathfrak{A}]$ **FO definable in each** $\mathfrak{A}$

relativises to natural fragments: $\forall^*$, $\exists^*$, FO^k, ML, GF, ...

boundedness as a decision problem

for a class $\mathcal{F}$ of FO formulae: $\mathrm{BDD}(\mathcal{F})$

given $\varphi(X, x) \in \mathcal{F}$
decide if $\mu_X \varphi$ is bounded

- **BDD a generalised SAT problem**: $(\varphi^{n+1} \wedge \neg \varphi^n)$ for all $n \in \mathbb{N}$
- **few decidable cases**, even for monadic recursion

primary motivation: detection of spurious fixed points

e.g. in Datalog, temporal logics, or circumscription

natural $\mathcal{F}$ are closed under iteration: $\forall^*$, $\exists^*$, FO^k, ML, GF, ...

decidability vs. undecidability for monadic BDD

undecidable	decidable
$\exists^*$ and even $\exists^*_+(\neq)$ **existential, positive with inequality** Gaifman, Mairson, Sagiv, Vardi 87	$\exists^*_+$ **pure existential positive** Cosmadakis, Gaifman, Kanellakis, Vardi 95
FO^2 **two variables** Kolaitis, O_ 98	ML **modal** O_ 98 and here
$\forall^*$ and even $\forall^*_-(=)$ **universal, mixed polarities or with equality** see proceedings	$\forall^*_-$ **universal, single polarities without equality** main result here

can encode tilings

decidable via tree codings

the rest of this talk:

(A) **new decidability argument for** $\mathrm{BDD}(\mathbf{ML})$
via reduction to MSO theory of ω-tree

(B) **reduction from** $\mathrm{BDD}(\forall^*_{-})$ **to** $\mathrm{BDD}(\mathbf{ML})$
based on ideas from guarded logic

(A) a new decidability argument for $\mathrm{BDD}(\mathbf{ML})$

ingredients:

- **Barwise–Moschovakis theorem for ML**:
  boundedness = ML-definability of fixpoint
- **characterisation of ML-definability**:
  ML-definability = $\sim$-invariance + tree-locality
- **tree-locility in MSO-theory of T_ω**:
  "König's lemma for T_ω"

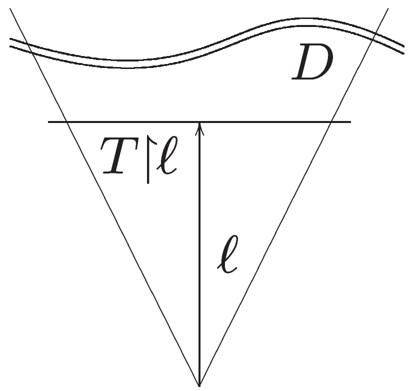
variations:
- ML systems
- restrictions to sufficiently rich MSO definable classes of trees

ML-definability = $\sim$-invariance + tree-locality

tree-locality (of $\sim$-invariant property ψ)

$\exists \ell \in \mathbb{N}$ such that for all ctble trees T,
and all initial $D \subseteq T$ with $D \supseteq T{\restriction}\ell$:

$$T \models \psi \quad \text{iff} \quad T{\restriction}D \models \psi$$



towards the MSO-theory of T_ω

$$\left.\begin{array}{l} Z \text{ initial and for all } \boldsymbol{I} \text{ and all initial } D\text{:} \\ Z \subseteq D \longrightarrow \big(\psi[\boldsymbol{I}] \leftrightarrow \psi[\boldsymbol{I}{\restriction}D]\big) \end{array}\right\} \; \eta(Z) \in \text{MSO}$$

ψ **tree-local** iff $T_\omega \models \exists Z \; \big(\boxed{Z \textbf{ bounded} \quad \text{not MSO}} \wedge \eta(Z)\big)$

König's lemma for regular expansions of T_ω

for regular (T_ω, Z) (finite number of subtrees up to $\simeq$)

t.f.a.e.:

(i) Z **path-finite** (no infinte path within Z)

(ii) Z **bounded** ($Z \subseteq T{\restriction}\ell$ for some $\ell \in \mathbb{N}$)

get tree-locality in MSO over T_ω**:**

$$
\begin{aligned}
T_\omega \models \exists Z\, \eta(Z) \;&\Leftrightarrow\; \exists Z \subseteq T_\omega \;\; (T_\omega, Z) \models \eta \\
&\Leftrightarrow\; (T_\omega, Z) \models \eta \text{ for some regular } (T_\omega, Z) \\
&\Leftrightarrow\; T_\omega \models \exists Z \big(Z \text{ bounded } \wedge\; \eta(Z) \big)
\end{aligned}
$$

(B) $\mathrm{BDD}(\forall^*_-)$ via reduction to modal case

3 main steps:

(1) Moschovakis normal form for universal induction

(2) guarded unfoldings
back & forth between relational structures and trees
faithful w.r.t. stages – but over elusive class of trees

(3) an MSO definable class of admissible trees
and a monotonicity argument just for $\forall^*_-$

(1) Moschovakis normal form for $\varphi(X,x) \in \forall^*$ (variant)

$\boldsymbol{\varphi(X,x) \;\equiv\; \forall y\big(\theta(x,y) \to \delta(X,y)\big)}$

$\delta = Xy_{i_1} \vee \ldots \vee Xy_{i_n}$

θ qfr-free, no X

θ positive for $\varphi \in \forall^*_-$

$$\boxed{\boldsymbol{\varphi(X,x) \;\equiv\; \bigwedge_i \forall y\big(\theta_i(x,y) \to \delta(X,y)\big)}}$$

θ_i conjunction of atoms

idea:
- $\boldsymbol{\theta_i}$ **as guards** w.r.t. new $R_i := \theta_i[\mathfrak{A}]$
- **unfold** expanded $\mathfrak{A}$ into tree-like structure
- **simulate** φ by system $\hat{\varphi}$ in ML

(2) unfolding and translation into ML

$$\mathfrak{A}^\theta \quad \longrightarrow \quad T(\mathfrak{A}^\theta) \quad \longrightarrow \quad (\mathfrak{A}^\theta)^* = \mathfrak{B}\big(T(\mathfrak{A}^\theta)\big)$$

tree structure

θ-guarded unfolding

$$\varphi(X,x) \quad \longrightarrow \quad (\widehat{\varphi}(\boldsymbol{X}))$$

system in ML

$$\boxed{\gamma[\varphi,\mathfrak{A}] = \gamma[\varphi,\mathfrak{A}^*] = \gamma[\widehat{\varphi},T(\mathfrak{A}^\theta)]}$$

but the class of these T
is not MSO definable

$$T \quad \longrightarrow \quad \mathfrak{B}(T)$$

consistent tree

represented by T

$$(\check{\varphi}(\boldsymbol{X},x)) \quad \longleftarrow \quad \varphi(X,x)$$

system in MSO

(3) monotonicity and admissible trees

compare systems $(\hat{\varphi}(\boldsymbol{X}))$ and $(\check{\varphi}(\boldsymbol{X},\boldsymbol{x}))$

in ML
faithful w.r.t. φ
over the $T(\mathfrak{A}^{\theta})$

in MSO
faithful w.r.t. φ
over all consistent T

by construction of $\mathfrak{B} = \mathfrak{B}(T)$, $\theta_i[\mathfrak{B}] \supseteq R_i^{\mathfrak{B}}$:

$\hat{\varphi}$ on T is an overestimate for φ on $\mathfrak{B}$ / $\check{\varphi}$ on T

admissible trees: consistent T with $\mu(\check{\varphi}) = \mu(\hat{\varphi})$ **MSO**

$\hat{\varphi}$ may still overestimate stages (go faster) but

φ bounded iff **$\hat{\varphi}$ bounded on admissible trees**

modal boundedness over rich MSO class of trees

some open questions

- $\mathrm{BDD}(\forall^*\_)$ for non-monadic recursion
- $\mathrm{BDD}(\mathbf{GF})$ (universal case essentially covered here)
- $\mathrm{BDD}(\mathbf{FO})$ for monadic recursion over bounded treewidth structures