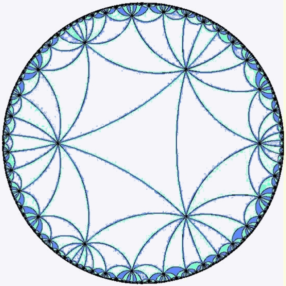

Highly Acyclic Groups Hypergraph Covers Guarded Fragment

Martin Otto TU Darmstadt

hyperbolic tesselation examples generated from sources at
http://aleph0.clarku.edu/~djoyce/poincare/poincare.html

motivation – from a guarded perspective

relational structure $\mathfrak{A}$	hypergraph of guarded subsets $H = H(\mathfrak{A})$	Gaifman graph: superposition of cliques $G(\mathfrak{A}) = G(H)$

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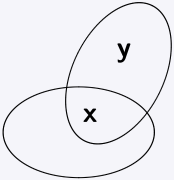
guarded fragment GF: quantification over guarded subsets/tuples governed by overlap pattern of $H(\mathfrak{A})$



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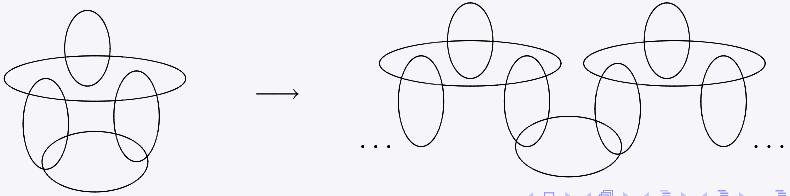

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not in FMT

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theorem (Andreka, van Benthem, Nemeti)

$\mathbf{GF} \equiv \mathbf{FO}/{\sim_{\mathrm{g}}}$ **also in FMT ??**

analogy: ML and GF / graphs and hypergraphs

graphs	**hypergraphs**
transition systems	relational structures
modal logic	guarded logic
$\mathbf{ML} \subseteq \mathbf{FO}$	$\mathbf{GF} \subseteq \mathbf{FO}$
bisimulation	**guarded bisimulation**
	hypergraph bisimulation
$\mathbf{ML} \equiv \mathbf{FO}/\!\sim$	$\mathbf{GF} \equiv \mathbf{FO}/\!\sim_{\mathbf{g}}$ **?? FMT**
acylicity: trees	tree-decomposable hypergraphs
N-local acyclicity in finite models	**??**

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acylicity: trees	tree-decomposable hypergraphs
N-local acyclicity in finite models	**N-acyclicity in finite hypergraphs**

basics: hypergraphs, hypergraph acyclicity

hypergraphs $\mathfrak{A} = (A, S)$ set of nodes A

set of hyperedges $S \subseteq \mathcal{P}(A)$

width $\mathrm{w}(\mathfrak{A}) = \max\{|s|: s \in S\}$

hypergraph acyclicity $=$ hypergraph tree decomposability

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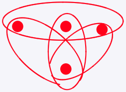
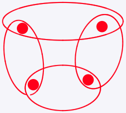


conformal: every clique covered by hyperedge



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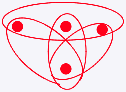

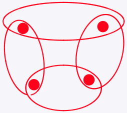

conformal: every clique covered by hyperedge

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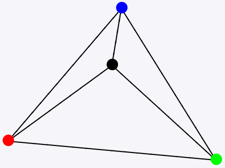
N-acyclic: relativisation to size N configurations

example: one of the simplest non-trivial hypergraphs

$\mathbf{H}_4^3$ the full width 3 hypergraph on 4 nodes;
$=$ tetrahedron with faces as hyperedges

irreducible: chordal but not conformal



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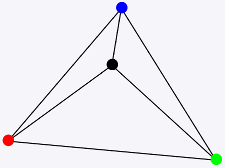
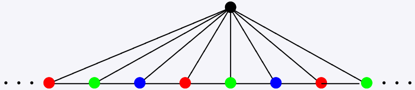


unfolds into acyclic hypergraph,
with typical 1-neighbourhood



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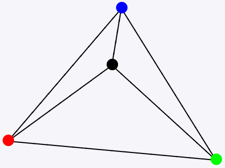

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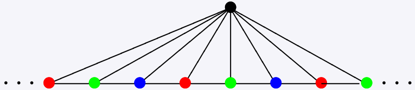
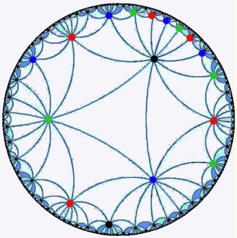


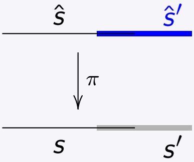
even 1-locally *infinite*,

or into *locally finite* hypergraph without *short* chordless cycles



definition: bisimilar (hypergraph) covers

$\pi : \hat{\mathfrak{A}} \xrightarrow{\sim} \mathfrak{A}$ locally bijective, strict homomorphism with *back* property w.r.t. hyperedges



(back): given s, s' and $\hat{s}$, can always find $\hat{s}'$

example: (guarded) tree unfolding over base structure

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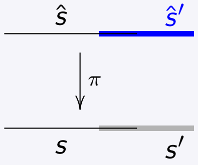

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question: **how much acyclicity in finite covers?**

key to relating GF to FO in FMT

definitions: degrees of acyclicity in (hyper)graph covers

the cover $\pi \colon \hat{\mathfrak{A}} \xrightarrow{\sim} \mathfrak{A}$ **is**

acyclic	if $\hat{\mathfrak{A}}$ is acyclic	
N-locally acyclic	if for all $\hat{a} \in \hat{\mathfrak{A}}$, $\ell \leqslant N$:	$\hat{\mathfrak{A}} \restriction \mathbf{N}^{\ell}(\hat{a})$ **acyclic**
N-acyclic	if for $\hat{B} \subseteq \hat{A}$, $\lvert\hat{B}\rvert \leqslant N$:	$\hat{\mathfrak{A}} \restriction \hat{\mathbf{B}}$ **acyclic**

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	forbidding small cyclic configurations!	

$\mathbf{H}^3_4$: even 1-locally acyclic *hypergraph* covers may be necessarily infinite

finite covers & limited forms of acyclicity

- **N-locally acyclic graph covers** (width 2) O_'02

  appl.: ML $\equiv$ FO/$\sim$ in non-classical contexts/FMT

- **conformal hypergraph covers** Hodkinson/O_'03

  appl.: fmp for GF $\longrightarrow$ fmp for CGF
  extns of Herwig–Hrushovski–Lascar EPPA

- **weakly N-acyclic covers** Barany/Gottlob/O_'10

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  PTIME canonisation for $\sim_{\mathrm{g}}$ & capturing PTIME/$\sim_{\mathrm{g}}$

- **N-acyclic covers** new here

  appl.: fmp for GF in classes with forbidden cyclic configs
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key ingredient: highly acyclic Cayley groups

groups without short cycles can be used

- in a straightforward product construction (graph case, width 2)
- in glueing of layers of local covers (hypergraph case, width > 2)

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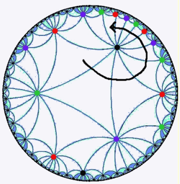
graph case: plain Cayley graphs of large girth (available)

hypergraphs: **need Cayley graphs without short cycles w.r.t. to a discounted distance measure (new)**

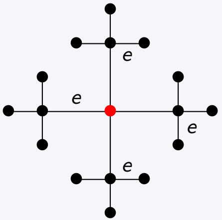
reason: sequences of hyperedge transitions may fix common node



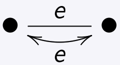
in a nut-shell: Cayley groups of large girth

find Cayley group $\mathbb{G}$ with involutive generators $e \in E$, of girth $> N$:

on regularly E-edge-coloured tree $\mathbf{T}$ of depth N,



let $e \in E$ operate through swaps of nodes in e-edges:



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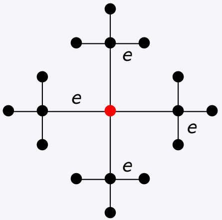

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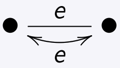

$$\mathbf{G := \langle E \rangle^{Sym(T)} \subseteq Sym(T)}$$
subgroup generated by the permutations $e \in E$

no short cycles: $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ for $k \leqslant N$

from ‘no short cycles’ to ‘no cycles with short colourings’

convention: $\alpha \subseteq E$: colour classes/generators of subgroups G_α

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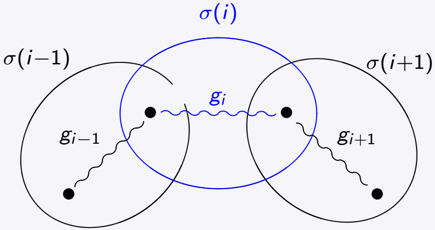
an n-colouring of some cycle in G is a tuple $(g_i)_{i \in \mathbb{Z}_n}$ s.t.

- $g_0 \circ \cdots \circ g_{n-1} = 1$
- f.a. i: $g_i \in G_{\sigma(i)}$ but $g_i \notin G_{\sigma(i-1)\cap\sigma(i)} \circ G_{\sigma(i)\cap\sigma(i+1)}$

non-trivial colour changes



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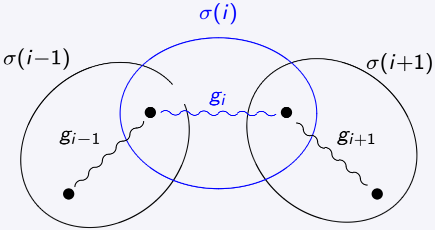

discounted distance measure:
n-colouring does not imply length bound!

theorem (N-acyclic groups)

For all k, $N \in \mathbb{N}$, there is a finite Cayley group with k involutive generators and without any n-coloured cycles for $n \leqslant N$.

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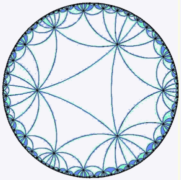
construction: inductively obtain groups

$$\{1\} = G^{(0)}, G^{(1)}, \dots, G^{(k)} = G$$

- for $|\alpha| \leqslant i$, $G^{(i)}_\alpha$ has no n-coloured cycles
- $G^{(i+1)}$ a subgroup of $\mathrm{Sym}(H)$ for E-coloured graph H obtained by amalgamation of copies of $G^{(i)}_\alpha$ for smaller α

in a nut-shell: N-acyclic covers

idea: use N-acyclic group to glue copies of some *locally finite* N-acyclic cover



along some peripheral finite boundary to mend defects without introducting new bad cycles/cliques

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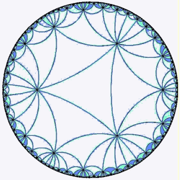

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theorem (N-acyclic covers)

For all finite $\mathfrak{A}$, $N \in \mathbb{N}$, there is a cover
$\pi : \hat{\mathfrak{A}} \xrightarrow{\sim} \mathfrak{A}$
by some *N*-acyclic and conformal $\hat{\mathfrak{A}}$.

model theory of conformal and N-acyclic structures

some special features

$N \gg n$ large enough:

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$N \gg n$ large enough:

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- similarly for 'direct connecting paths' = short chordless paths
- and even uniform size bounds for

finitary closure operation w.r.t. 'direct connecting paths'

FMT characterisation of GF $\qquad$ GF $\equiv$ FO$/\sim_{\mathrm{g}}$ (FMT)

want: Ehrenfeucht–Fraïssé upgrading $\sim_{\mathrm{g}}^{\ell} \longrightarrow \equiv_{\mathbf{q}}$

$$
\begin{tikzcd}
\mathfrak{A} \arrow[rr, dash, "\sim_{\mathrm{g}}^{\ell}"] \arrow[d, "\sim_{\mathrm{g}}"] & & \mathfrak{B} \arrow[d, "\sim_{\mathrm{g}}"] \\
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what for?

this yields: $\varphi \in \mathsf{FO}_q$ invariant under $\sim_{\mathrm{g}}$

$\Rightarrow$ φ invariant under $\sim_{\mathrm{g}}^{\ell}$

$\Rightarrow$ $\varphi \equiv \varphi' \in \mathsf{GF}_{\ell}$ **classically & FMT**

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\Big| & & & & \Big| \\
\sim_{\mathrm{g}} & & & & \sim_{\mathrm{g}} \\
\downarrow & & & & \downarrow \\
\mathfrak{A}^* & \text{——} & \begin{array}{c}\sim_{\mathrm{g}}^{\ell} \\ \equiv_q\end{array} & \text{——} & \mathfrak{B}^*
\end{array}
$$

how to?

pass to **guarded covers** $\quad \mathfrak{A}^* \ \sim_{\mathrm{g}}^{\ell} \ \mathfrak{B}^*$

suitably saturated w.r.t. multiplicities and **sufficiently acyclic** to imply $\mathfrak{A}^* \ \equiv_q \ \mathfrak{B}^*$

summary — finite covers & limited forms of acyclicity

- **N-locally acyclic graph covers** O_'02

  appl.: ML $\equiv$ FO/$\sim$ in non-classical contexts/FMT

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  PTIME canonisation for $\sim_g$ & capturing PTIME/$\sim_g$

  optimal complexity!

- **N-acyclic covers** new here

  appl.: fmp for GF in classes with forbidden cyclic configs
  GF $\equiv$ FO/$\sim_g$ in FMT

  maximal acyclicity!