Bisimulation Invariance and Instances of Expressive Completeness

Oviedo 2003

Martin Otto
University of Wales Swansea → TU Darmstadt
www-compsci.swan.ac.uk/~csmartin

preservation theorems

classical examples (within FO):

semantics	$\longleftrightarrow$	**syntax**
preservation under extensions	$\longleftrightarrow$	existential
monotonicity in P	$\longleftrightarrow$	positive in P

$\boxed{\leftarrow}$ **preservation** simple/via syntax

$\boxed{\rightarrow}$ **expressive completeness**

for suitable φ^*
establish $\varphi^* \equiv \varphi$ via
preservation of φ

model theory proper

invariance under an equivalence

$\leftrightharpoons$ invariance of φ

$$\mathfrak{A} \leftrightharpoons \mathfrak{B} \quad \Rightarrow \quad \Big[\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi\Big]$$

- φ expresses $\leftrightharpoons$ invariant property
- φ does not distinguish between $\leftrightharpoons$ equivalents
- **φ is about $\leftrightharpoons$-classes**

interest in modelling:

want structures up to $\leftrightharpoons$	need logic that is $\leftrightharpoons$ invariant

a prime example

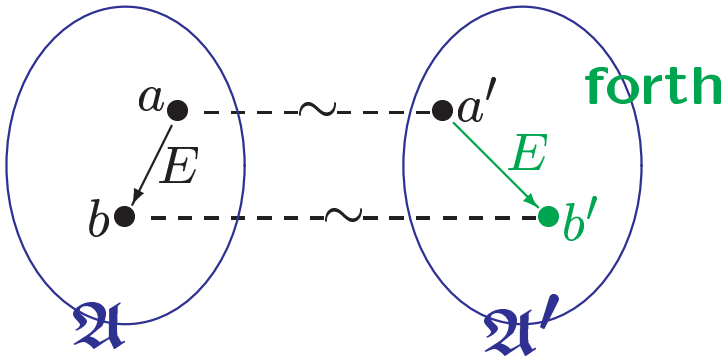
bisimulation equivalenece $\sim$
$\sim$ classes: process behaviours

process logics are the $\sim$ invariant logics

other important notions of equivalence
based on back-and-forth games like $\sim$

bisimulation games, $\sim$

back & forth along edges



$Z : \mathfrak{A} \sim \mathfrak{A}'$

- $\emptyset \neq Z \subseteq A \times A'$
- respects unary predicates
- back & forth along edges

variations: two-way and / or global bisimulations
in particular $\approx$: **two-way and global**

bisimulation games as **Ehrenfeucht-Fraïssé games** for corresponding modal logics

bisimulation games, $\sim$ and $\sim^\ell$

bisimulation equivalence:	strategy for player II in infinite game	$\sim$
ℓ-bisimulation equivalence:	strategy for player II in ℓ-round game = equivalence in modal logic ML up to nesting depth ℓ, $\equiv_\ell^{\mathsf{ML}}$	$\sim^\ell$

ML: basic modal logic

atomic propositions (unary predicates)
closure under Booleans, and
modal quant. $\Box/\Diamond$ $\qquad \forall y(Exy \to _) \;/\; \exists y(Exy \wedge _)$

[extensions: inverse and global modalities; ML$[-, \forall]$ etc]

preservation (straight from Ehrenfeucht-Fraïssé analysis)

$\varphi \in \mathbf{ML}_\ell$ **invariant under** $\sim^\ell$ [similarly for variants]

characterisation (van Benthem's Theorem)

$\mathbf{FO}/{\sim} \equiv \mathbf{ML}$

equivalent for $\varphi(x) \in \mathbf{FO}$:

- φ **is** $\sim$ **invariant**
- φ **is equivalent to** $\varphi^* \in \mathbf{ML}$

$\mathbf{FO}/{\sim} \equiv \mathbf{ML}$ van Benthem

- characterising ML $\subseteq$ FO
- effective syntax for FO/$\sim$
- **ML is the first-order logic of process behaviour**

similarly, and more importantly for applications,
at the level of MSO

$\mathbf{MSO}/{\sim} \equiv \mathbf{L}_{\boldsymbol{\mu}}$ Janin/Walukiewicz

- $\mathbf{L}_{\boldsymbol{\mu}}$ **is the monadic SO logic of process behaviour**

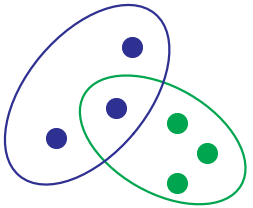
examples of other back&forth equivalences

with extra-logical meanings in terms of "restricted access"

guarded bisimulation equivalence

patches covered by ground atoms

"qualitative access restriction"

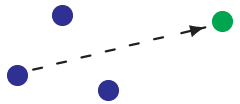


$\mathbf{FO}/\sim_{\mathbf{g}} \equiv \mathbf{GF}$ Andreka/van Benthem/Nemeti

k-pebble equivalence

up to k elements at a time

"quantitative access restriction"



$\mathbf{FO}/\simeq^{k} \equiv \mathbf{FO}^{k}$ Immerman/Kozen
Barwise/van Benthem

emphasis on methodological points

- proof techniques
- variations (exploring robustness)
- classical versus finite model theory

the fmt picture is different

for instance $\mathbf{FO}/\simeq^2 \not\equiv \mathbf{FO}^2$ (FIN)

but $\mathbf{FO}/\sim \equiv \mathbf{ML}$ (FIN) Rosen

NB: $\sim$ invariance (FIN) $\not\Rightarrow$ $\sim$ invariance

what's typical about $\sim/\sim^\ell$

full equivalence	approximants	of finite index
$\leftrightharpoons$	$\leftrightharpoons^\ell$	
infinite games	**finite, ℓ-round games**	

Ehrenfeucht-Fraïssé analysis of $\leftrightharpoons^\ell$

$\longrightarrow$ approximants to full characterisation thm

$$\mathbf{FO}/\leftrightharpoons^\ell \;\equiv\; \mathcal{L}_\ell \quad \text{as in} \quad \mathrm{FO}/\sim^\ell \;\equiv\; \mathrm{ML}_\ell$$

full characterisation thm equivalent to **compactness property**

$\sim$ **invariance** $\Rightarrow$ $\sim^\ell$ **invariance for some** ℓ

classical proofs: compactness of FO

based on convergence $\leftrightharpoons^{\ell} \longrightarrow \leftrightharpoons$

in $*$-models (e.g., ω saturated) where $\leftrightharpoons^{\omega} := \bigcap_{\ell} \leftrightharpoons^{\ell}$ is $\leftrightharpoons$

for $\leftrightharpoons$ invariant φ:

$\mathfrak{A}_\ell$	$\leftrightharpoons^{\ell}$	$\mathfrak{B}_\ell$	(one ℓ at a time)
$\mathfrak{A}$	$\leftrightharpoons^{\omega}$	$\mathfrak{B}$	(all ℓ simultaneously)
$\mathfrak{A}^*$	$\leftrightharpoons$	$\mathfrak{B}^*$	
$\top\!\!\top$		$\top\!\!\top$	#
$\mathfrak{G}$		$\mathfrak{G}$	

non-constructive (indirect)
does not go through in fmt

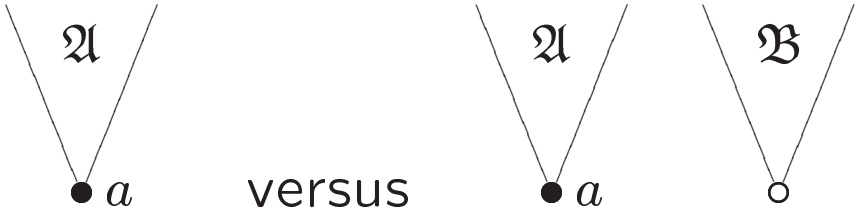
illustration
(new) stand-alone proof for van Benthem-Rosen

reduces input from classical model theory to Ehrenfeucht-Fraïssé
$\longrightarrow$ valid classically as well as in fmt

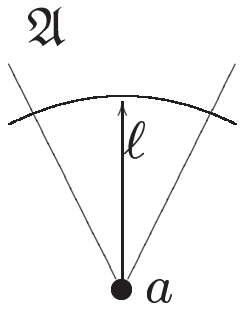
(0) $\varphi \sim$ **invariant** $\Rightarrow$ φ **invariant under disjoint unions**

(1) $\varphi \sim$ **invariant** $\Rightarrow$ φ ℓ**-local for** $\ell \leqslant 2^{\mathrm{qr}(\varphi)}$ **(E-F)**

(2) $\varphi \sim$ **invariant &** ℓ**-local** $\Rightarrow$ $\varphi \sim^{\ell}$ **invariant**



invariance under disjoint union

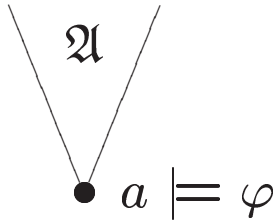


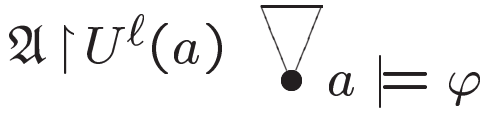
ℓ-locality

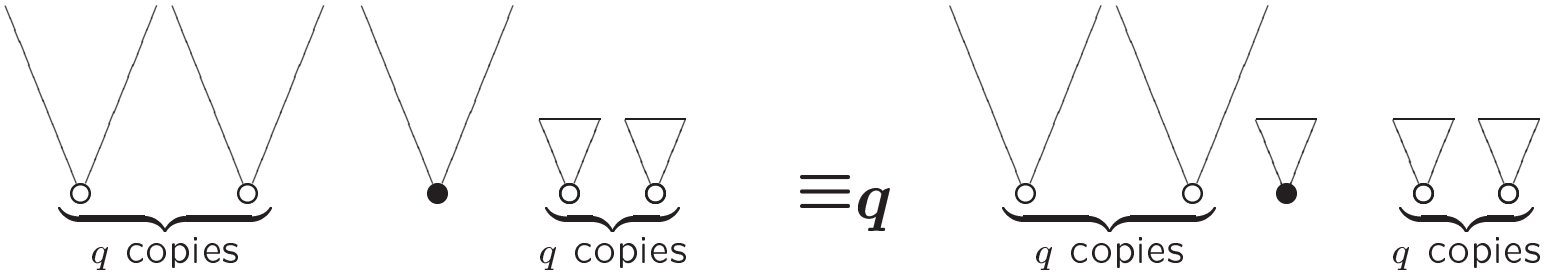
the Ehrenfeucht-Fraïssé argument

$\varphi \sim$ invariant $\Rightarrow$ φ ℓ-local for $\ell = 2^{\mathrm{qr}(\varphi)}$

show $\boxed{\mathfrak{A}, a \models \varphi \quad \textbf{iff} \quad \mathfrak{A} \restriction U^\ell(a), a \models \varphi}$





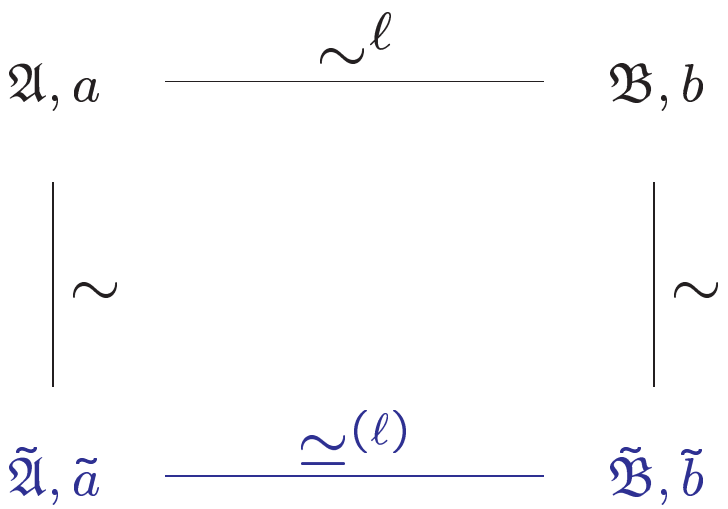


play q rounds respecting critical distance $d_m = 2^{q-m}$ in round m

$\sim$ inv. & ℓ-local $\Rightarrow$ $\sim^\ell$ inv. (almost trivial here)

upgrading $\sim^\ell$ to $\simeq^{(\ell)}$

$\simeq$ of ℓ-neighbourhoods



partial unravelling to depth ℓ
+ adjustment of multiplicities

$\simeq^{(\ell)}$ preserves ℓ-local φ

$\varphi \sim$ invariant and ℓ-local $\Rightarrow$ $\sim^\ell$ invariant

robustness/variations/extensions

finer equivalences

over finite/general transition systems

- global bisimulation (non-trivial locality)
- two-way bisimulation
- guarded bisimulation

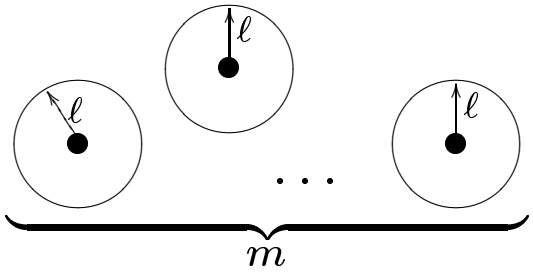
other classes of (finite or infinite) frames

- especially: connectivity constraints

what follows: some ingredients / results / apparent limitations

Gaifman locality

$\equiv^{\ell}_{q,m}$ agreement w.r.t. ℓ-local properties of quantifier rank q in up to m many disjoint ℓ-neighbourhoods



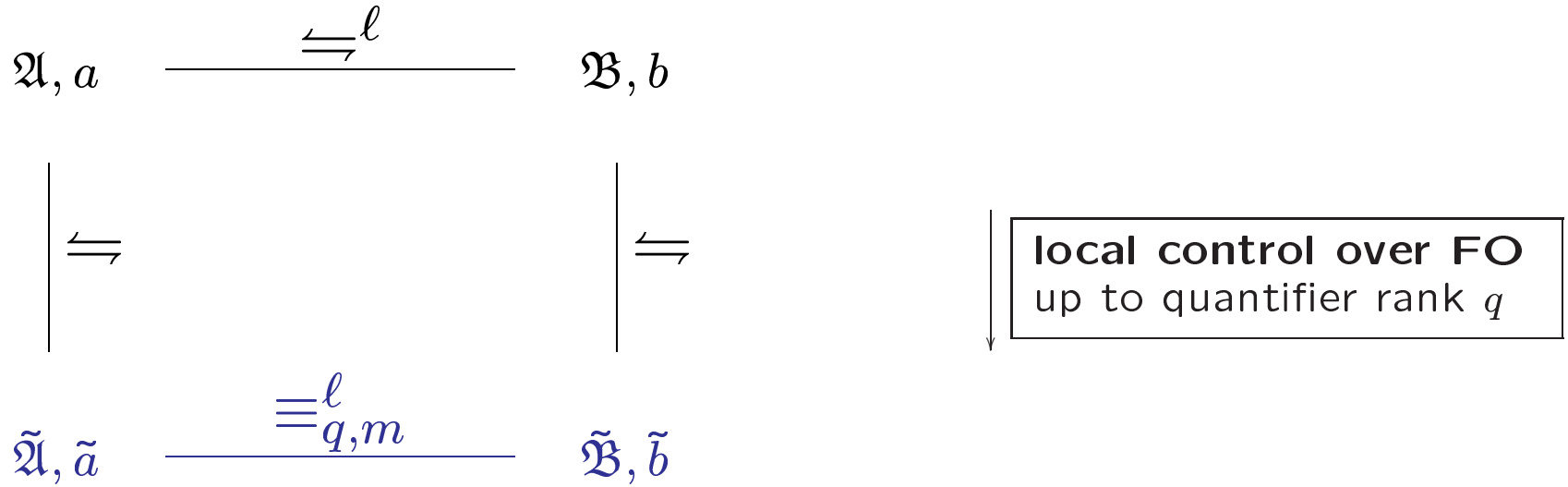
Gaifman's Theorem

any first-order $\varphi(x)$ preserved by some $\equiv^{\ell}_{q,m}$

$\rightarrow$ use $\equiv^{\ell}_{q,m}$ as finite index approximation to $\equiv$

local control over FO suffices

generic idea: upgrading $\leftrightharpoons^\ell$ to $\equiv^\ell_{q,m}$



$\varphi(x)$ preserved under $\equiv^\ell_{q,m}$ and $\leftrightharpoons$ invariant

$\Rightarrow$ φ $\leftrightharpoons^\ell$ invariant

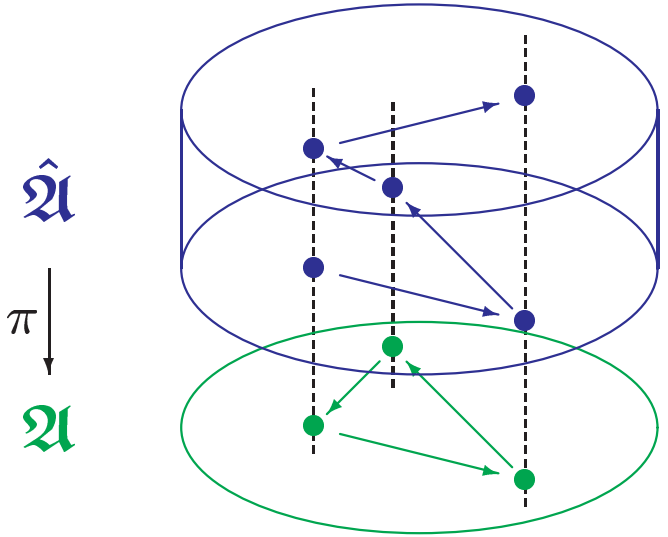
works for . . .		results
		classically and in fmt
$\sim_\forall$	global (forward) bisimulation	$\mathbf{FO}/{\sim_\forall} \equiv \mathbf{ML}[\forall]$
$\approx$	global two-way bisimulation	$\mathbf{FO}/{\approx} \equiv \mathbf{ML}[-,\forall]$
$\sim_\mathbf{g}$	guarded bisimulation	$\mathbf{FO}/{\sim_\mathbf{g}} \equiv \mathbf{GF}$ (over transition systems)

from $\leftrightharpoons^{\ell}$ to local control over FO

locally acyclic covers

instead of (infinite) tree unravellings

homomorphism $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$
whose graph induces a
two-way global bisimulation



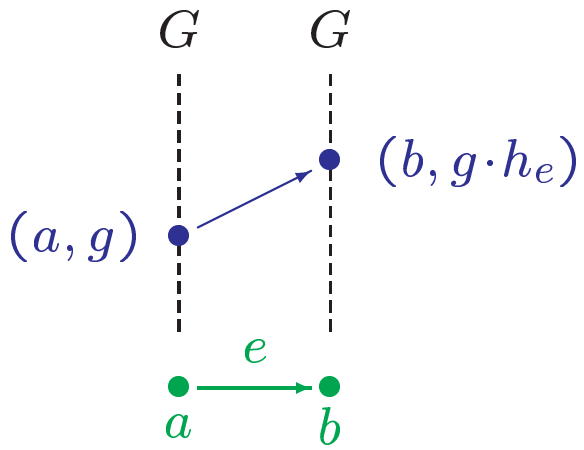
two-way unravellings are (infinite) acyclic covers

theorem

any [finite] transition system admits a $\approx$ cover by a [finite] ℓ-locally acyclic transition system.

proof: "fibre bundle" over base system
using group whose Cayley graph
has no short cycles

[polynomial blow-up for fixed ℓ]



robustness / variations

- works in fmt sense
- can deal with mismatches like $\sim_\forall$ versus $\equiv^\ell_{\dots}$
  directional symmetric
- works in other restricted classes of interest

  notably: connectivity constraints

  $\mathbf{FO}/\sim \;\equiv\; \mathbf{ML}[\forall] \quad (\mathbf{conn})$

  $\sim$-type of finite transition system not even ML[$\forall$] definable over disconnected frames

limitations

the limits of locality

most notable: $\mathbf{MSO}/\sim$ and $\mathbf{L}_\mu$

MSO has no sense of locality
e.g. can quantify over paths

status of Janin/Walukiewicz

$\mathbf{MSO}/\sim \equiv \mathbf{L}_\mu$

OPEN in **FMT**

similarly $\mathbf{FO}/\sim_{\mathbf{g}} \equiv \mathbf{GF}$ in fmt (arbitrary relational)

some partial results — joint work with Colin Hirsch

bisimulation invariance $\Rightarrow$ tree model property

unravellings

finite systems $\leftrightarrow$ finitely branching regular trees

FREG

theorem (Hirsch/O; unpubl.)

for all $\varphi \in$ MSO: **$\sim$ invariance over FREG $\Rightarrow$**
$\sim$ invariance over all trees

via (non-trivial) pruning

indications towards $\mathbf{MSO}/\sim \equiv \mathrm{L}_\mu$ (FIN) ?

(A) consider $\mathbf{MSO}^*$: MSO with CTL* like semantics in terms of unravellings

corollary

$\mathbf{MSO}^*/\sim \equiv \mathrm{L}_\mu$ (FIN)

$\mathbf{MSO}^{\mathrm{path}}/\sim \equiv \mathbf{CTL}^*$ (FIN) (cf. Moller/Rabinovich)

(B) consider closure of L_μ within MSO/$\sim$ (FIN)

corollary

for $\varphi = \exists \boldsymbol{X} \psi(\boldsymbol{X})$, $\psi(\boldsymbol{X}) \in \mathrm{L}_\mu$: **$\sim$ invariance over FIN**
$\Rightarrow$ **full $\sim$ invariance**

other generalisations of the Janin/Walukiewicz result

theorem (Grädel/Hirsch/O)

$\mathbf{GSO}/\sim_{\mathbf{g}} \equiv \mu\mathbf{GF}$

guarded second-order logic / guarded fixed-point logic

theorem (Grädel/O; unpubl.)

$\mathbf{MSO}/\sim_{-} \equiv \mathbf{L}_{\mu}[-]$

two-way bisimulation / full mu-calculus