Methods for Deciding Boundedness of Least Fixed Points

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compare:

b child of a, Rab	versus	b descendant of a, R^*ab
diameter $\leqslant 17$		diameter $< \infty$, connectivity
depth < 17		well-foundedness
static		**dynamic**

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diameter $\leqslant 17$		diameter $< \infty$, connectivity
depth < 17		well-foundedness
static		**dynamic**
FO		**not FO** but expressible using **least fixed points** of monotone, monadic relational FO recursion

least fixed points — monotone relational recursion

fixed point extensions $\left\{ \begin{array}{lll} \mathbf{FO} & \longrightarrow & \mathbf{LFP} \\ \mathbf{ML} & \longrightarrow & \mathbf{L}_\mu \\ \mathbf{GF} & \longrightarrow & \mu\mathbf{GF} \end{array} \right.$

massive boost in expressiveness

LFP : all Ptime properties of ordered finite structures (Immerman/Vardi)
L_μ : all bisimulation invariant MSO properties of finite transition systems (Janin–Walukiewicz)

least fixed points — monotone relational recursion

(monadic) least fixed point induction
on $\boldsymbol{\varphi(X, x)}$, positive in X

$\varphi(X, x)$ induces *monotone* operation on subsets

$$\begin{array}{rcl} \varphi\colon \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A) \\ P & \longmapsto & \varphi[\mathfrak{A}, P] := \{a \in A\colon \mathfrak{A} \models \varphi[P, a]\} \end{array}$$

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with unique **least fixed point** $\boxed{(\mu_{\mathsf{X}}\varphi)[\mathfrak{A}] = \bigcup_\alpha \mathsf{X}^\alpha[\mathfrak{A}]}$

generated from **inductive stages**

$$\begin{aligned} \mathsf{X}^0[\mathfrak{A}] &= \emptyset \\ \mathsf{X}^{\alpha+1}[\mathfrak{A}] &= \varphi[\mathfrak{A}, \mathsf{X}^\alpha[\mathfrak{A}]] \\ \mathsf{X}^\lambda[\mathfrak{A}] &= \bigcup_{\alpha<\lambda} \mathsf{X}^\alpha[\mathfrak{A}] \end{aligned}$$

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stage/rank of $a \in \mu_X\varphi[\mathfrak{A}]$: $\min_{\alpha}(a \in X^{\alpha}[\mathfrak{A}])$

least fixed points – monotone relational recursion

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stage/rank of $a \in \mu_X\varphi[\mathfrak{A}]$: $\min_\alpha(a \in X^\alpha[\mathfrak{A}])$

depth of φ-recursion on $\mathfrak{A}$:

closure ordinal $\gamma[\varphi, \mathfrak{A}] = \min_\alpha\big(\mathbf{X}^{\alpha+1}[\mathfrak{A}] = \mathbf{X}^\alpha[\mathfrak{A}]\big)$

unbounded in general across all structures

boundedness

$\varphi(X, x)$ bounded: $\exists n \in \mathbb{N}$ s.t. $\gamma[\varphi, \mathfrak{A}] < n$ for all $\mathfrak{A}$

$\varphi(X, x)$ bounded on class $\mathcal{C}$, analogous

boundedness a highly non-trivial semantic property
of formula specifying a recursive process

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boundedness a highly non-trivial semantic property
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boundedness as a decision problem

for a class $\mathcal{F}$ of X-positive formulae (and class $\mathcal{C}$ of structures):

BDD($\mathcal{F}$) / **BDD($\mathcal{F}$, $\mathcal{C}$)**

given $\varphi(X, x) \in \mathcal{F}$
decide if φ is bounded / bounded over $\mathcal{C}$

very few decidable cases, even for monadic recursion

for $\mathcal{F}$ with natural closure properties:

- **φ bounded $\Rightarrow$ $\mu_{\mathbf{X}}\varphi$ uniformly $\mathcal{F}$-definable:**
  finite stages definable by substitution-iterates $\varphi^{\mathbf{n}}(\mathbf{x}) \in \mathcal{F}$

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- **SAT reducible to BDD:**
  guard unbounded $\varphi(X, x)$ by ψ (relativised for non-interference)

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  compare SAT for $(\varphi^{n+1} \wedge \neg\varphi^{n})_{n\in\mathbb{N}}$

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- **φ unbounded $\Rightarrow$ all finite increments can be non-trivial**
  for $\varphi \in$ FO compare SAT for $\bigwedge_n(\varphi^{n+1} \wedge \neg\varphi^{n})$

(with compactness even get $\gamma[\varphi, \mathfrak{A}] = \omega$; essential towards B–M thm)

boundedness and definability

Barwise–Moschovakis theorem (BM 78)

for any X-positive FO formula $\varphi(X, x)$
the following are equivalent:

(i) **φ bounded**

(ii) **$\mu_X\varphi$ uniformly FO definable**

(iii) **$\mu_X\varphi[\mathfrak{A}]$ FO definable in each $\mathfrak{A}$**

relativises to natural fragments: $\forall^*$, $\exists^*$, FO^k, ML, GF, ...

relativises to elementary/projective classes:
acyclic, treewidth k, ...

compactness!

undecidability vs. decidability for monadic BDD within FO

undecidable	decidable
$\exists^*$ and even $\exists^*_+(\neq)$ **existential, positive** **with inequality** Gaifman, Mairson, Sagiv, Vardi 87	$\exists^*_+$ [Datalog] **pure existential positive** Cosmadakis, Gaifman, Kanellakis, Vardi 95
FO^2 **two variables** Kolaitis, O_ 98	ML **modal** O_ 98, improved 06
$\forall^*$ and even $\forall^*_-(=)$ **universal, mixed polarities** **or with equality** O_ 06	$\forall^*_-$ **pure universal negative** O_ 06

can encode tilings in grids

decidable via tree codings

example: decidability via MSO on trees, for BDD(ML)

core: Barwise–Moschovakis & locality + MSO-coding in trees

recall Barwise–Moschovakis for modal fixed points:

$\varphi(\mathsf{X}) \in \mathbf{ML}$ **bounded** $\Leftrightarrow$ $\mu_{\mathsf{X}}\varphi \in \mathbf{ML}$

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by bisimulation invariance & Löwenheim–Skolem:

- restrict attention to (countable) tree-models
- over trees, capture ML-definability by a locality criterion
- crux: to get locality criterion into MSO

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by bisimulation invariance & Löwenheim–Skolem:

- restrict attention to (countable) tree-models
- over trees, capture ML-definability by a locality criterion
- crux: to get locality criterion into MSO
- over regular trees, capture ML-definability by MSO-definable locality criterion

$\longrightarrow$ **reduction to Rabin's decidability for MSO over trees**

decidability of BDD(ML) and its wider ramifications

decidability proofs
based on Barwise–Moschovakis (FO-definability of $\mu_X \varphi$)
and (generalised) locality arguments in trees

- modulo some pre-processing the above idea essentially lifts to deciding BDD($\forall^*_-$) (O_ LICS 06)

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- with much more sophisticated Gaifman locality arguments:

theorem (Kreutzer, O_, Schweikardt ICALP 07)

BDD(FO, $\mathcal{AC}$) decidable
for the class $\mathcal{AC}$ of all acyclic graph structures

limitations

Barwise–Moschovakis couples boundedness to definability

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at a cost: restriction to elementary classes

e.g., neither applicable to the class of all trees
nor the class of all finite acyclic graph structures

End of Part I

$\mathsf{BDD}(\mathcal{F}, \mathsf{all})$ for interesting fragments $\mathcal{F}$	versus	**$\mathsf{BDD}(\mathsf{FO}, \mathcal{C})$** for interesting classes $\mathcal{C}$

$\mathsf{BDD}(\mathsf{FO}, \mathcal{AC})$

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note potential explanatory power w.r.t. apparent dichotomy

undecidable BDD	**decidable BDD**
grids and tilings	tree-like models

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$\rightarrow$ look to **"generalised tree model property for BDD"**
to explain all known classical decidable cases, & new

nice-model-properties for BDD

BDD($\mathcal{F}$) has the $\mathcal{C}$-model-property
if for all $\varphi(X, x) \in \mathcal{F}$:
φ bounded $\Leftrightarrow$ φ bounded over $\mathcal{C}$

behaviour on $\mathcal{C}$
indicative for BDD

in this case, decidabilty of BDD($\mathcal{F}, \mathcal{C}$)
implies decidabilty of BDD($\mathcal{F}$) = BDD($\mathcal{F}, \mathrm{all}$)

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interesting candidates:

- $\mathcal{C} = \mathcal{FIN}$ (finite model property for BDD): **ML**, $\exists^*_+$, $\forall^*_-$
- $\mathcal{C} = \mathcal{T}$ (tree model property for BDD): **ML**, $\mathbf{L}_\mu$
- $\mathcal{C} = \mathcal{T}_k$ (btw model property for BDD): **ML**, $\exists^*_+$, $\forall^*_-$, **GF**, $\mathbf{L}_\mu$, μ**GF** (!)

another leap — from BDD(FO, $\mathcal{C}$) to BDD(MSO, $\mathcal{C}$)

(Blumensath, O_, Weyer ICALP 09) & ongoing

new approach via

MSO coding and automata

divorcing boundedness/definability: Barwise–Moschovakis lost

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key ingredients/ideas:

- coding of fixpoint histories in X-positive MSO-types
- consistent history annotations of $\mathfrak{A}, a$ $(\mathfrak{A} \in \mathcal{C},\ a \in (\mu_X \varphi)[\mathfrak{A}])$ recognised by automaton $\mathcal{A}_\varphi$
- stage of $a \in \mu_X \varphi[\mathfrak{A}]$ corresponds to minimal weight of accepting run of $\mathcal{A}_\varphi$ as a *distance automaton*

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$\longrightarrow$ $\boxed{\text{reduction of BDD(MSO, } \mathcal{C}) \text{ to}}$ *limitedness* problems for distance automata $\mathcal{A}_\varphi$ on $\mathcal{C}$

some key ideas in sketches

- X-positive types
- histories of X-positive types
- extraction of stage succession from annotation
- stages "counted" by distance automata

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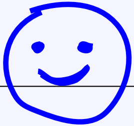
really: some key ideas in over-simplified sketches



X-positive MSO-m-type in variables X, x:

$$\mathbf{t}^{\mathbf{m}}(\mathfrak{A}, \mathsf{P}, \mathsf{a}) = \{\psi(\mathsf{X}, \mathsf{x}) \in \mathrm{MSO}^{\mathbf{m}}(\mathsf{X}^{+}) \colon \mathfrak{A} \models \psi[\mathsf{P}, \mathsf{a}]\}$$

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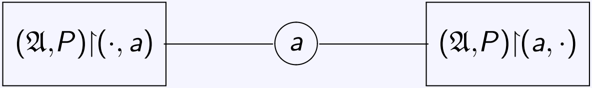
governed by (monotone) MSO-composition rules
e.g., in a string graph:



$$t^m(\mathfrak{A}, P, a) =$$

$$t^m((\mathfrak{A},P){\restriction}(\cdot, a)) \oplus t^m((\mathfrak{A},P){\restriction}\{a\}, a) \oplus t^m((\mathfrak{A},P){\restriction}(a, \cdot))$$

(1) X-positive types

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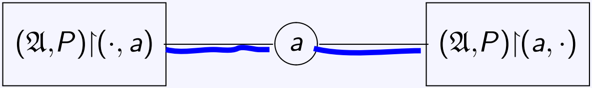

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---

basic idea: **annotate $\mathbf{a} \in \mathfrak{A}$ with $\big(\mathbf{t}^{\mathbf{m}}(\mathfrak{A}, \mathbf{X}^{\alpha}, \mathbf{a})\big)_{\alpha \leqslant \gamma[\varphi, \mathfrak{A}]}$**

(2) histories of X-positive types

annotation of $\mathfrak{A}, a$ by a history tiling (here: in string graph)

finite alphabet of tiles $\begin{cases}\text{each representing a snapshop} \\ \text{of information flow through } x\end{cases}$

$[x, \cdot)$	$(\cdot\,,\cdot)$	$(\cdot\,, x]$	**out**
			$\Uparrow$
$(\cdot\,, x)$	$[x]$	$(x, \cdot)$	**in**

left **right**

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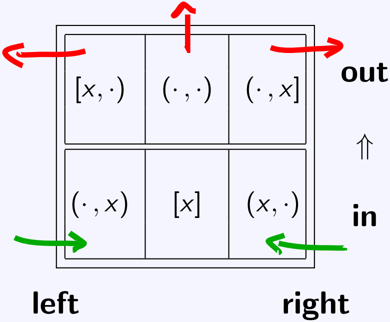


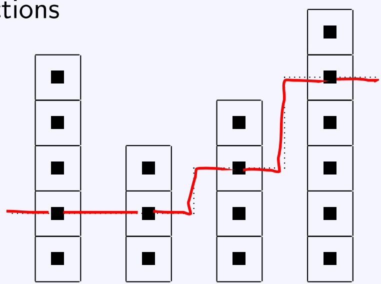
vertical stacking of tiles: succession of stages

horizontal matches: communication with neighbours
with MSO-composition rules as local consistency conditions

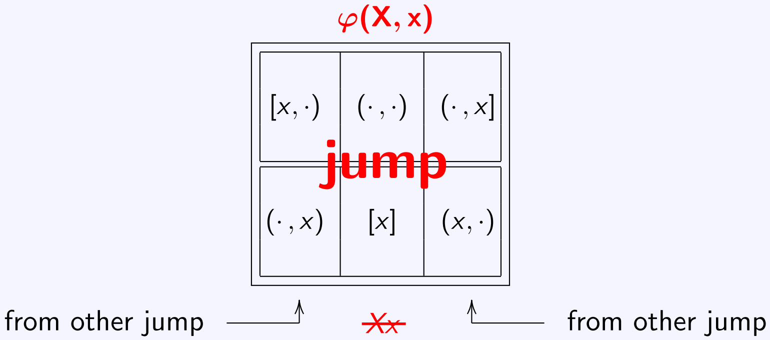
(3) partial extraction of stage succession from sections

finite alphabet, finite no. of tiles	$\Rightarrow$	histories recorded without duplicates	$\Rightarrow$	desynchronisation

can only reconstruct approximations through synchronisation along consistent sections



(4) stages from dependent sequences of jumps



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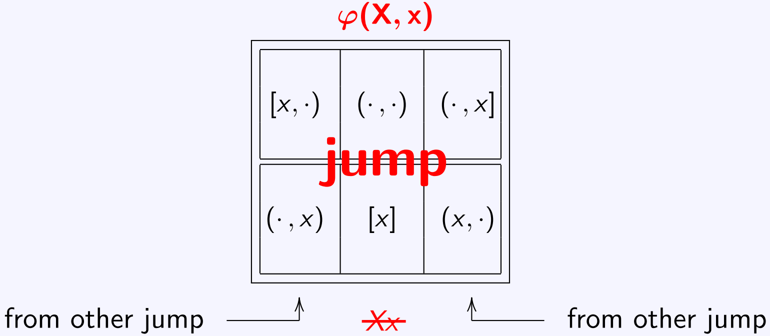

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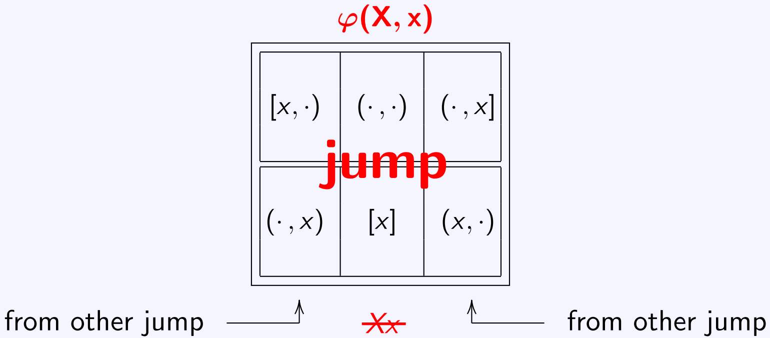

- $\boldsymbol{\gamma[\varphi,\mathfrak{A}]}$ bounded by lengths of sequences of dependent jumps

- use **distance automata** to $\begin{cases} \text{check consistency of annotation} \\ \text{count lengths of jump sequences} \\ \text{to marked } a \in \mu_X\varphi[\mathfrak{A}] \end{cases}$

reduction to limitedness of distance automata

distance automaton $\mathcal{A}_\varphi$ over $\mathcal{C}$

accepting all consistent annotations of $\mathfrak{A}, a$

with $\mathfrak{A} \in \mathcal{C}$, $a \in \mu_X \varphi[\mathfrak{A}]$ s.t.

weights of accepting runs *are* **lengths of jump sequences :**

minimal weight of annotation of $\mathfrak{A}, a$	$\approx$	length of shortest jump sequence	$\approx$	stage of $a \in \mu_X \varphi[\mathfrak{A}]$

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then

L($\mathcal{A}$) limited iff φ bounded on $\mathcal{C}$

(weight bounded)

decidability of limitedness of distance automata:

(A) NFA on finite words (Hashiguchi 90)

(B) automata on finite trees (Colcombet–Löding CSL 08)

(C) parity automata on infinite trees (Colcombet–Löding)
announced 09

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consequences for decidability of BDD

BDD(MSO, $\mathcal{C}$) decidable over these classes $\mathcal{C}$:

(A) **finite string graph structures** (BOW ICALP 09) finite words

(B) **finite acyclic graph structures** finite trees

(C) **acyclic graph structures** trees

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BDD(MSO, $\mathcal{C}$) decidable over these classes $\mathcal{C}$:

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by robustness of BDD(MSO, $\mathcal{C}$) under MSO interpretations and nice-model-properties for BDD:

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(2) covers GSO over *finite structures of fixed finite tree width*, yields decidability of **BDD($\exists^*_+$)** and **BDD($\forall^*_-$)**

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(1) covers GSO over *finite structures of fixed finite path width*

(2) covers GSO over *finite structures of fixed finite tree width*, yields decidability of **BDD($\exists^*_+$)** and **BDD($\forall^*_-$)**

(3) covers GSO over *all structures of fixed finite tree width*, yields decidability of **BDD(FO, $\mathcal{AC}$)** (KOS 07)
and of **BDD(GF)**, **BDD($\mathbf{L}_\mu$)**, **BDD(μGF)** (new)

summary

from case-to-case to a rationale behind perceived dichotomy

Barwise–Moschovakis & **locality:**
goes some way to explain key positive results
where BDD $= \mathrm{definability}$

MSO & **distance automata:**
goes much further in explanation of
combinatorial/graph theoretic dichotomy

albeit away from definability (and logic?)

+ several new BDD decidability results

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The End

extras:

- **decidability via locality in trees: BDD(ML)**
- **undecidability via dominoes: BDD($\forall^*$)**

example: decidability via MSO on trees, for BDD(ML)

core: Barwise–Moschovakis & locality + MSO-coding in trees

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by bisimulation invariance & Löwenheim–Skolem:

- restrict attention to (countable) tree-models
- over trees, capture ML-definability by a locality criterion
- crux: to get locality criterion into MSO

example: decidability via MSO on trees, for BDD(ML)

core: Barwise–Moschovakis & locality + MSO-coding in trees

recall Barwise–Moschovakis for modal fixed points:

$\varphi(\mathbf{X}) \in$ **ML bounded** $\Leftrightarrow$ $\mu_{\mathbf{X}}\varphi \in$ **ML**

by bisimulation invariance & Löwenheim–Skolem:

- restrict attention to (countable) tree-models
- over trees, capture ML-definability by a locality criterion
- crux: to get locality criterion into MSO
- over regular trees, capture ML-definability by MSO-definable locality criterion

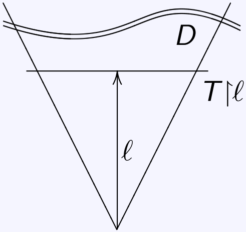
$\longrightarrow$ **reduction to Rabin's decidability for MSO over trees**

dfn: tree-locality of $\psi \in$ MSO

$\exists \boldsymbol{\ell} \in \mathbb{N}$ such that
for all trees T and for all
initial $D \subseteq T$ with $D \supseteq T{\restriction}\ell$:

$T \models \psi$ iff $T{\restriction}D \models \psi$

semantics only depends on
bounded initial segment



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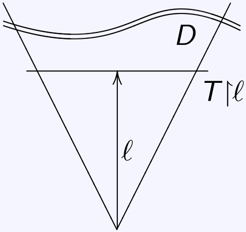

for bisimulation invariant $\psi \in$ MSO:

$$\begin{array}{rl} & \psi(x) \text{ tree-local (with radius } \ell) \\ \Leftrightarrow & \psi(x) \text{ expressible in ML (at nesting depth } \ell) \end{array}$$

with modal Barwise–Moschovakis:
$\Rightarrow$ locality-testing for $\psi = \mu_X \varphi$ decides boundedness of $\varphi(X, x)$

(tree-locality of $\psi \in$ MSO) $\in$ MSO ?

$\exists \boldsymbol{\ell} \in \mathbb{N}$ such that
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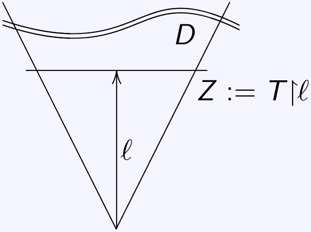

$$\left.\begin{array}{l} Z \text{ initial and for all } \mathbf{I} \text{ and all initial } D\text{:} \\ Z \subseteq D \longrightarrow \big(\psi[\mathbf{I}] \leftrightarrow \psi[\mathbf{I}{\upharpoonright}D]\big) \end{array}\right\} \ \boldsymbol{\eta}(\mathbf{Z}) \in \mathbf{MSO}$$

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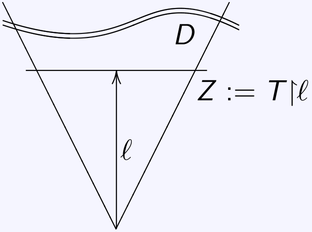

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$\boldsymbol{\psi}$ **tree-local** iff $\mathbf{T}_\omega \models \exists \mathbf{Z} \ \big(\boxed{\mathbf{Z} \text{ bounded}} \wedge \ \boldsymbol{\eta}(\mathbf{Z})\big)$

not MSO

König's lemma for regular expansions of $\mathbf{T}_\omega$

for regular $(\mathbf{T}_\omega, \mathbf{Z})$ (regular: finite no. of subtrees up to $\simeq$)
with initial $Z \subseteq T_\omega$ t.f.a.e.:

(i) **Z path-finite** (no infinite path within Z)

(ii) **Z bounded** ($Z \subseteq T\restriction\ell$ for some $\ell \in \mathbb{N}$)

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corollary

tree-locality decidable for $\psi(x) \in$ MSO

hence: **BDD(ML) decidable**

in fact, the inclusion "ML $\subseteq$ L$_\mu$" is thus decidable

example: undecidability via tiling for BDD($\forall^*$)

reduction of the tiling problem for tiling systems $\mathcal{D}$
to (un)boundedness of $\varphi^{\mathcal{D}}(X, x)$ in $\forall^*_=$ with equality

$\varphi^{\mathcal{D}}(X, x) = \varphi_0^{\mathcal{D}} \wedge \varphi_1(X, x)$

$\varphi_0^{\mathcal{D}}$: H and V the graphs of commuting partial functions, colours $(P_d)_{d \in \mathcal{D}}$ compatible with tiling constraints

$\varphi_1(X, x)$ s.t. $a \notin X^n[\mathfrak{A}] \;\Rightarrow\; \exists\, h \colon \begin{array}{ccc} (n \times n)\text{-grid} & \xrightarrow{\text{hom}} & (A, H, V) \\ (0,0) & \longmapsto & a \end{array}$

then $\quad \mathcal{D}$ tiles $\mathbb{N} \times \mathbb{N}$-grid

$\Leftrightarrow \quad \mathcal{D}$ tiles arbitrarily large $(n \times n)$-grids $\qquad$ König's lemma

$\Leftrightarrow \quad \varphi^{\mathcal{D}}$ unbounded

example: undecidability via tiling for BDD($\forall^*$) ctd.

φ_1 can be chosen in universal ML $\subseteq \forall^*_-$:

$$\boldsymbol{\varphi_1(X) := \Box_H X \;\vee\; \Box_V X \;\vee\; \Box_H \Box_V X \;\vee\; \Box_V \Box_H X}$$

$a \notin \varphi_1^{n+1}[\mathfrak{A}]$

$\mathfrak{A} \models \varphi_0$

$\Rightarrow$

$$\begin{tikzcd}
a_{01} \arrow[r, "H"'] & a_{11} \\
a \arrow[u, "V"'] \arrow[r, "H"'] & a_{10} \arrow[u, "V"']
\end{tikzcd}$$

for some $a_{10}, a_{01}, a_{11} \notin \varphi_1^n[\mathfrak{A}]$

overlapping homomorphisms h_{ij} of $(n \times n)$-grids at a_{ij} glued to get a homomorphism of $((n+1) \times (n+1))$-grid at a

compatibility guaranteed by φ_0