Symmetry-Preserving Finite Synthesis & Amalgamation

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Simons Institute
Berkeley, Nov. 2016

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- **amalgamation: overlap specifications**
- **local vs. global consistency**
- **local vs. global symmetry**

finite model constructions with a touch of combinatorial geometry and group theory

amalgamation?

- **synthesis & amalgamation:**
  hypergraphs & relational structures: (de)composition patterns
  model theory & algebra: homogeneity, Fraïssé limits
  geometry: atlases of charts & changes of co-ordinates

- **local/partial symmetries:**
  logic & model theory: partial isomorphisms
  geometry & algebra: groupoids, inverse semigroups

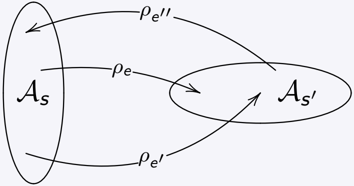
- **local consistency:**
  logic & combinatorics: constraint satisfaction
  logic & databases: constraints, dependencies
  logic & computation: e.g. quantum information

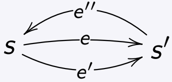
. . . **calling for global solutions, closures, completions**

a generic free amalgamation construction

from finite families

$$\left.\begin{array}{rl} (\mathcal{A}_s)_{s \in S} & \text{of relational structures} \\ (\rho_e \colon \mathcal{A}_s \xrightarrow{\text{part}} \mathcal{A}_{s'})_{e \in E[s,s']} & \text{of partial isomorphisms} \end{array}\right\} (*)$$





$\mathrm{I} = (S, E)$

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find natural free amalgam of disjoint copies
$(\mathcal{A}_s, w) \simeq \mathcal{A}_s$ tagged by walks w in $\mathrm{I} = (S, (E[s, s']))$,
with $(\mathcal{A}_s, w)$ and $(\mathcal{A}_{s'}, w \cdot e)$ overlapping according to ρ_e

$$\big(((\mathcal{A}_s), (\rho_e)) \otimes \mathrm{I}^* \big) \;/\; \approx$$

based on multi-sorted monoid I^* of walks in I

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based on multi-sorted monoid I^* of walks in I

- realises the amalgamation pattern $(*)$
- is free in a universal algebraic sense
- is generic but infinite (unless I is acyclic)

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based on multi-sorted monoid I^* of walks in I

. . . finite analoga?
. . . based on what instead of I^* ?
. . . equally homogeneous?
. . . of controlled acyclicity?

realisation of $\mathbb{H} = \big((\mathcal{A}_s), (\rho_e)\big)$ in general:

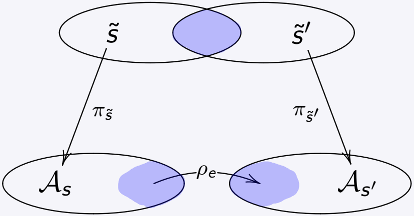
a relational structure $\mathcal{A}$ with an atlas given by superimposed hypergraph structure $(A, \tilde{S})$ of charts $\pi_{\tilde{s}} \colon \mathcal{A}{\restriction}\tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$ s.t.

- **locally, all ρ_e-overlaps are realised:**

  each $\pi_{\tilde{s}}^{-1}(\mathcal{A}_s)$ overlaps with some $\pi_{\tilde{s}'}^{-1}(\mathcal{A}_{s'})$ according to ρ_e

- **globally, no incidental overlaps occur:**

  if $\tilde{s} \cap \tilde{t} \neq \emptyset$, then this is due to a composition of ρ_{e_i} for some *single* walk $w = e_1 \cdots e_m$ from $\pi(\tilde{s})$ to $\pi(\tilde{t})$



realisation of $\mathrm{H} = \big((\mathcal{A}_s), (\rho_e)\big)$ in general:

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NB: the second, "no-nonsense" condition avoids potential relational inconsistencies for amalgams

aside: the rôle of groupoids/inverse semigroups

composition structure of partial bijections:

- with partial composition (as a total operation)
  $\rightsquigarrow$ inverse semigroups
- with exact composition (as a partial operation)
  $\rightsquigarrow$ groupoids

groupoids capture local/partial symmetries

. . . just as groups capture global symmetries

(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}\big)$ with

associative compositions $G_{st} \times G_{tu} \to G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

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the potentially finite analogue of the infinite path monoid I^*

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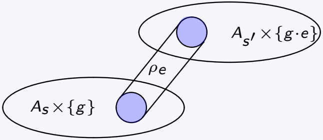
$\rightsquigarrow$ **reduced products $\mathbf{H} \otimes \mathbb{G}$**
as candidate realisations

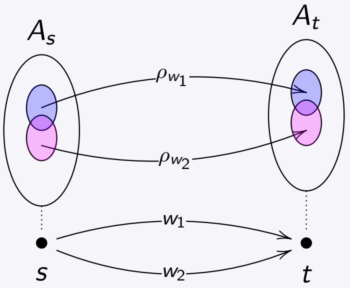


when is this a realisation of $\mathbf{H}$?

obstructions: wrong kind of cycles (1)

violation of path independence for elements

- conflicting identifications may collapse individual A_s



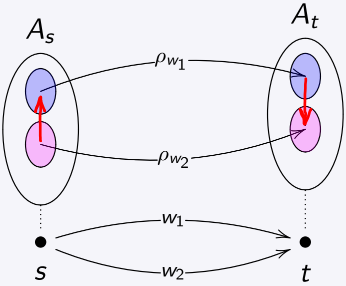
a bad cycle?

can be overcome by relatively simple pre-processing:
replace I by finite covering Î that unfolds bad cycles

obstructions: wrong kind of cycles (2)

violations of path independence for tuples

- incidental parallel identifications may cause conflicts at the relational level

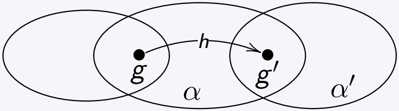


a bad cycle?

$\rightsquigarrow$ **substantial acyclicity conditions on suitable groupoids**

an appropriate notion of acyclicity: control coset cycles

- not just short generator cycles in the Cayley graph of $\mathbb{G}$, but short cycles of *cosets* $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



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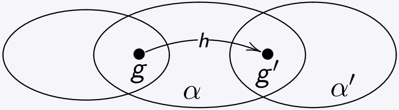
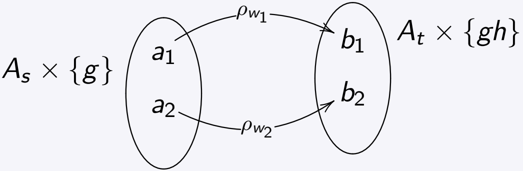
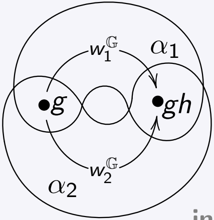


- in particular, need to avoid certain coset cycles of length 2





$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$

in reduced product

in $\mathbb{G}$

bad coset cycles & obstruction no. (2)

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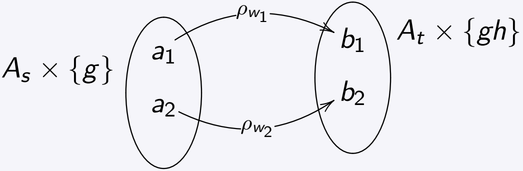

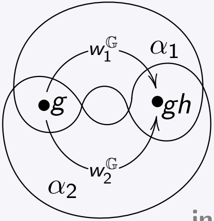

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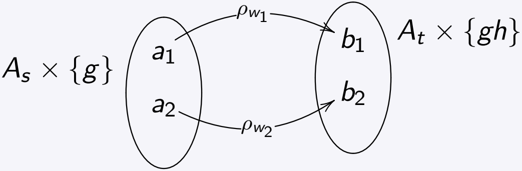

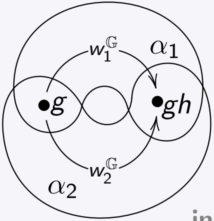

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in reduced product **in $\mathbb{G}$**

α_1: generators carrying a_1
α_2: generators carrying a_2 } want $h = w^{\mathbb{G}}$ for $w \in \langle \alpha_1 \cap \alpha_2 \rangle$ so that w carries both a_1 *and* a_2

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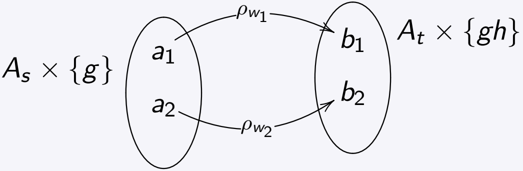

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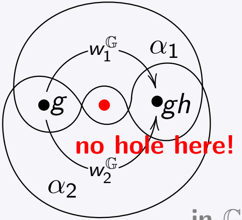

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$$\langle \boldsymbol{\alpha_1} \rangle \cap \langle \boldsymbol{\alpha_2} \rangle = \langle \boldsymbol{\alpha_1} \cap \boldsymbol{\alpha_2} \rangle$$

avoiding a coset 2-cycle

any degree of acyclicity in finite groupoids

theorem (O_13)

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without coset cycles of length up to N

any degree of acyclicity in finite groupoids

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idea: in an inductive construction generate $\mathbb{G}$ from (semi)group action on amalgamation chains that unfold short cosets cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles

any degree of acyclicity in finite groupoids

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corollary

for any $\mathrm{H} = \big((\mathcal{A}_s), (\rho_e)\big)$ obtain realisations $\mathrm{H} \otimes \mathbb{G}$ that

- respect all symmetries of H (genericity)
- have any desired degree of (local) acyclicity

with an atlas of charts onto the $(\mathcal{A}_s)$
forming an N-acyclic hypergraph
with changes of co-ordinates generated by the (ρ_e)

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generic & locally free finite realisations
of any finite amalgamation pattern

applications (1): hypergraph coverings

corollary

every finite hypergraph admits, for $N \in \mathbb{N}$, finite coverings that

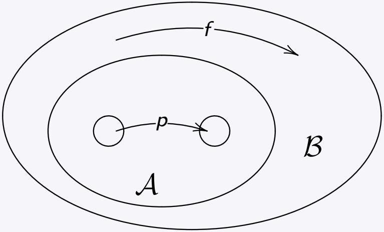
- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable, α-acyclic)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

idea: generic *N*-acyclic finite realisations of the overlap pattern corresponding to the "exploded view" of the given hypergraph

applications (2): lifting local to global symmetries

EPPA: extension properties for partial automorphisms

Hrushovski 92 (graphs)
Herwig 98 (relational structures)
Herwig–Lascar 00 (fmp w.r.t. forbidden homomorphisms)



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here, as a corollary

for finite $\mathcal{A}$ and collection of partial automorphisms (ρ_p) find finite EPPA extension $\mathcal{B} \supseteq \mathcal{A}$ as generic realisation of pattern of self-overlaps of $\mathcal{A}$ induced by the ρ_p

- with atlas of $\mathcal{A}$-charts forming N-acyclic hypergraph on B
- hence, locally free & universal w.r.t. homomorphisms

which implies full Herwig–Lascar result

applications (3): various uses in modal/guarded logics

- **finite model properties & finite controllability for guarded logics and constraints**

  using finite coverings of controlled acyclicity and/or Herwig–Lascar extension properties

- **characterisation theorems** (fmt & classical) **for the guarded fragment GF and relatives**

  using finite coverings of controlled acyclicity

- **characterisation theorems** (fmt & classical) **for (modal) common knowledge logic**

  new, with Felix Canavoi, using (finite) $S5$-frames over Cayley groups with controlled coset-acyclicity

some related references

Bárány–Gottlob–O_(LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

Bárány–ten Cate–O_(VLDB 2012, arXiv:1203.0077): Queries with guarded negation

Grädel–O_(2014): The freedoms of (guarded) bisimulation

Hodkinson–O_(BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

Herwig–Lascar (Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

O_(Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

O_(arXiv:1404.4599): Finite groupoids, finite coverings and symmetries in finite structures

→ `http://www.mathematik.tu-darmstadt.de/~otto/`