Integer Points in Polyhedra

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Exercise Sheet 1

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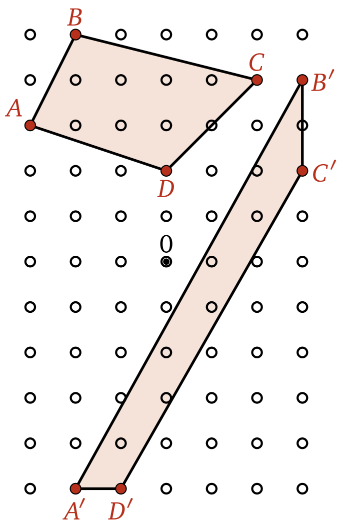

1.1. Let P be the lattice polygon with the vertices A, B, C, D (and P' respectively with the vertices A', B', C', D') as given in the figure to the right.

(1) Compute the areas of P and P'.

(2) Are P and P' isomorphic? If yes, find an explicit unimodular transformation mapping P to P'.

You may want to explore maps of the form

$$x \longmapsto \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} x$$

for some $k \in \$Z$ and translations for this.

1.2. Show that an integral matrix $A \in \mathbb{Z}^{d\times d}$ has an integral inverse if and only if $\det A = \pm 1$, that is,

$$\mathrm{Gl}_d(\mathbb{Z}) = \left\{A \in \mathbb{Z}^{d\times d} \mid \det A = \pm 1\right\}.$$

1.3. Let P be a lattice polygon with i interior lattice points, b boundary lattice points and volume a. Prove that

$$a = i + \frac{b}{2} - 1.$$

1.4. A vector $v \in \mathbb{Z}^2$ (or in any lattice) is called *primitive* if it is not a non-trivial integer multiple of some other lattice vector.

Show that any primitive $v \in \mathbb{Z}^2$ is part of a lattice basis.

1.5. Let P be a lattice polygon, $b(P)$ the number of boundary lattice points, and $i(P)$ the number of interior lattice points.

Prove that a given pair (b, i) of nonnegative integers equals $(b(P), i(P))$ for some lattice triangle P if and only if there exist integers $p, q, r \in \mathbb{Z}$ with $p > 0$ and $0 \le q < r$ such that

$$b = p + \gcd(q, r) + \gcd(q - p, r) \qquad \text{and} \qquad i = \frac{pr - b}{2} + 1.$$

In this case, the triangle with vertices $(0,0), (p,0), (q,r)$ can be chosen.

1.6. In the lecture we proved that equivalent lattice polygons have the same number of lattice points, the same total length of the edges and the same volume. Show that the converse is not true in general.