Integer Points in Polyhedra

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Exercise Sheet 3 Summer 19
April 26, 2019

1.1. Let P be a lattice polygon with one interior lattice point, nine boundary lattice points and volume $9/2$. Prove that P is the triangle $3\Delta_2$ (up to unimodular transformation).

Hint: You may assume that we are in the setting of the case we left out in the proof of Scott's Theorem: P is squeezed into a box of height 3 and the intersection with the upper and lower side of the box has total length 3.

1.2. By Scott's Theorem all lattice polygons with volume a, $i \geq 1$ interior and b boundary lattice points are either unimodularly equivalent to $3\Delta_2$ or satisfy the inequlities

$$i \geq 1 \qquad\qquad b \geq 3 \qquad\qquad b \leq 2(i+3)\,.$$

Show that any pair (i, b) that satisfies these ineqalities comes from a lattice polygon.

1.3. Let $n \geq 1$ and

$$D_n := \left\{ x \in \mathbb{Z}^d \mid \sum_{i=1}^{d} x_i \text{ is even} \right\}.$$

Show that D_n and, for even n, also $D_n^+ := D_n \cup \left(D_n + \frac{1}{2}\mathbb{1}\right)$ are lattices.

1.4. Let V be a vector space and $B := \{b_1, \ldots, b_k\} \in V$ linearly independent. Show that

$$\Lambda(B) := \left\{ \sum_{i=1}^{k} \lambda_i b_i \mid \lambda_i \in \mathbb{Z} \right\} \qquad (*)$$

is a lattice.

1.5. Let $B := \{b_1, \ldots, b_d\} \in V$ be a basis of a vector space V, $\Lambda(B)$ as in (*) and

$$\Pi(B) := \left\{ \sum_{i=1}^{k} \lambda_i b_i \mid 0 \leq \lambda_i < 1 \right\}.$$

Show that any $u \in V$ has a unique representation as $u = a + v$ for $a \in \Lambda$ and $v \in \Pi(B)$.

1.6. Let $\Lambda \subset \mathbb{R}^d$ be a lattice of rank d and

$$\Lambda^* := \{\alpha \in (\mathbb{R}^d)^* \mid \alpha(u) \in \mathbb{Z} \text{ for all } u \in \Lambda\}\,.$$

Λ^* is the *dual lattice* to Λ.

Let $u_1, \ldots, u_d$ be a basis of Λ and $\alpha_1, \ldots, \alpha_d$ the corresponding dual basis (i.e. $\alpha_i(u_j) = 1$ if $i = j$, and $\alpha_i(u_j) = 0$ otherwise).

(1) Prove that Λ^* is spanned by $\alpha_1, \ldots, \alpha_d$ as a lattice.
(2) Prove that $\Lambda^{**} = \Lambda$.
(3) Prove that $\det(\Lambda)\det(\Lambda^*) = 1$.

1.7. Euler's Formula states that a finite planar graph with v nodes, e edges and f bounded faces satisfies

$$v - e + f = 1$$

Show that this is equivalent to Pick's Formula.