Integer Points in Polyhedra

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Exercise Sheet 3 | Summer 19
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∘ 3.1. Let Λ' be a sublattice of Λ and rank $\Lambda = d$. Then there is a lattice basis $\mathbf{u}_1, \dots, \mathbf{u}_d \in \Lambda$ of Λ and $s_1, \dots, s_d \in \mathbb{Z}_{\geq 0}$ with $s_1 | s_2 | \dots | s_d$ such that $s_1\mathbf{v}_1, \dots, s_k\mathbf{v}_k$ is a basis of Λ', where $k = \max(i \mid s_i \neq 0)$.

Deduce that for a basis $\mathbf{u}_1, \dots, \mathbf{u}_d$ of Λ' we have

$$\left|\Lambda/\Lambda'\right| = \left|\Pi(\mathbf{u}_1, \dots, \mathbf{u}_d) \cap \Lambda\right| = \frac{\det \Lambda'}{\det \Lambda}.$$

Hint: For the first part write the bases in coordinates with respect to Λ and use the Smith normal form $S = LAR$ in the transformation $AR = L^{-1}S$.

∘ 3.2. Show that any lattice polytope P has a regular triangulation using only the vertices of P.

Recall: A *subdivision* $\mathcal{T}$ of a polytope P is a pure polyhedral complex such that the union of the cells is P. The set $V(\mathcal{T})$ of vertices of $\mathcal{T}$ is the union of the vertices all its cells. $\mathcal{T}$ is a *triangulation* if all cells are simplices.

$\mathcal{T}$ is *regular* if there is a weight function $w : V(\mathcal{T}) \to \mathbb{R}$ such that all cells are obtained as the projection onto the first d coordinates of the lower hull of

$$Q := \operatorname{conv}\left((\mathbf{v}, w(\mathbf{v})) : \mathbf{v} \in V\right),$$

where the *lower hull* are those facets of Q whose facet normal has negative last coordinate.

3.3. Give an example that shows that not all triangulations of a lattice polytope are regular.

∘ 3.4. Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope (or any convex body) Show that there is a simplex $S \subseteq P$ with vertices $\mathbf{v}_0, \dots, \mathbf{v}_d$ such that

$$P \subseteq (-d)(S - \mathbf{x}) + \mathbf{x} = (-d)S + (d+1)\mathbf{x} \tag{1}$$

and

$$P \subseteq (d+2)(S - \mathbf{x}) - \mathbf{x} = (d+2)S - (d+1)\mathbf{x} \tag{2}$$

where

$$\mathbf{x} := \frac{1}{d+1}\sum_{i=0}^{d} \mathbf{v}_i$$

is the centroid of S.

Hint: Choose $S := \operatorname{conv}(\mathbf{v}_0, \dots, \mathbf{v}_d) \subseteq P$ with maximal volume in P.

For any $0 \leq i \leq d$ let H_i be the facet hyperplane of the facet of S not containing $\mathbf{v}_i$, $r_i := \operatorname{dist}(\mathbf{v}_i, H_i)$ and $R_i := \{\mathbf{x} : \operatorname{dist}(\mathbf{x}, H_i) \leq r_i\}$.

Show that $P \subseteq R_i$ for $0 \leq i \leq d$.

Express $\bigcap_{i=0}^{d} R_i$, $(-d)(S - \mathbf{x}) + \mathbf{x}$, and $(d+2)(S - \mathbf{x}) - \nu x$ in barycentric coordinates with respect to $\mathbf{v}_0, \dots, \mathbf{v}_d$ and compare.

⋆ 3.5. Let $\Gamma \subseteq \mathbb{R}^d$ be a lattice or rank d. Construct a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ and a face F where

$$\operatorname{aff} F \cap \Lambda \neq \operatorname{aff} F \cap \Lambda(P \cap \Gamma) \neq \operatorname{aff} F \cap \Lambda(F \cap \Gamma).$$

⋆ 3.6. Show that any subdivision S of a polygon P such that $\mathsf{V}(\mathsf{S}) = \mathsf{V}(P)$ is regular.

3.7. Finish the exercises of Sheet 2.