Integer Points in Polyhedra

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Exercise Sheet 4 | Summer 19

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∘ 4.1. For a field $\mathbb{k}$ let $\mathsf{L} := \mathbb{k}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ be the *Laurent polynomial ring* of polynomials whose monomials have the form $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdot x_2^{a_2} \cdots x_m^{a_m}$ for some $a_1, a_2, \dots, a_m \in \mathbb{Z}$. We consider the L-module $\mathsf{L}^{\mathsf{f}} := \mathbb{k}[\![t_1^{\pm 1}, \dots, t_d^{\pm 1}]\!]$ of *formal Laurent series*, *i.e.* formal (possibly infinite) sums of Laurent monomials.

A formal series $G \in \mathsf{L}^{\mathsf{f}}$ is *summable* if there are $g, f \in \mathsf{L}$ such that $f := gG$. Let $\mathsf{L}^{\mathrm{sum}}$ be the set of summable series.

(a) Show that $\mathsf{L}^{\mathrm{sum}}$ is an L-submodule of L^{f}, *i.e.*, show that for $f \in \mathsf{L}$ and $G, H \in \mathsf{L}^{\mathrm{sum}}$ also $f \cdot G$ and $G + H$ are summable.

(b) Let $R := \mathbb{k}(x_1, \dots, x_m)$ be the field of rational functions of L and define a map

$$\begin{aligned} \phi : \mathsf{L}^{\mathrm{sum}} &\longrightarrow R \\ G &\longmapsto f/g\,. \end{aligned}$$

Show that this is a homomorphism from summable series to rational functions, *i.e.*, show that for $f \in \mathsf{L}$ and $G, H \in \mathsf{L}^{\mathrm{sum}}$ we have $\phi(f \cdot G) = f\phi(G)$ and $\phi(G + H) = \phi(G) + \phi(H)$.

∘ 4.2. Let $C := \operatorname{cone}(\mathbf{v}_1, \dots, \mathbf{v}_m) \subseteq \mathbb{R}^d$ be a polyhedral cone. We say that a vector $\xi \in \mathbb{R}^d$ is *generic* with respect to C if ξ is not in the linear hull of any $(d-1)$-dimensional face of C.

The *half-open cone* C^ξ with respect to a generic ξ is

$$C^\xi \;:=\; \{\mathbf{y} \in C \;:\; \mathbf{y} + \varepsilon\xi \in C \text{ for all } \varepsilon > 0 \text{ small enough}\}\,.$$

(a) Show that

$$C^\xi \;=\; \left\{\mathbf{y} \in C \;:\; (1-\varepsilon)\mathbf{y} + \varepsilon \in C \text{ for all } \varepsilon > 0 \text{ small enough}\right\}.$$

In plain words, the half-open cone contains all faces *not* visible from ξ.

(b) Show that, if C is simplicial and ξ generic, then in the unique representation $\xi = \sum \lambda_i \mathbf{v}_i$ all coefficients are non-zero and

$$C^\xi \;:=\; \left\{ \sum \mu_i \mathbf{v}_i \;:\; \mu_i \geq 0 \text{ for } i \in I_+(x) \text{ and } \mu_i > 0 \text{ for } i \in I_-(x) \right\}$$

for $\qquad I_+(\xi) \;:=\; \{i \;:\; \lambda_i > 0\} \qquad$ and $\qquad I_-(\xi) \;:=\; \{i \;:\; \lambda_i < 0\}$.

(c) If $\xi \in C$ and generic, then $C^\xi = C$ and $C^{-\xi} = \operatorname{int} C$.

Now let $\mathscr{T}$ be a triangulation of a polyhedral cone C and ξ generic with respect to all cones in $\mathscr{T}$. Let $\mathscr{T}_d$ be the set of maximal cones in $\mathscr{T}$.

(d) Show that we have a *disjoint* union

$$C^{\xi} = \bigsqcup_{D \in \mathscr{T}_d} D^{\xi}.$$

In particular

$$C = \bigsqcup_{D \in \mathscr{T}_d} D^{\xi} \qquad \text{and} \qquad \operatorname{int} C = \bigsqcup_{D \in \mathscr{T}_d} D^{-\xi}.$$

4.3. Let C be a pointed *affine* polyhedral cone, *i.e.* there is $\mathbf{t} \in \mathbb{Z}^d$ such that $C' := C - \mathbf{t}$ is a pointed polyhedral cone with apex in the origin.

(a) Write down an integer point generating function for C using one for C'.

(b) Do this explicitely for the cones $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \leq 3\}$.

(c) Add the two generating functions and rewrite them as a series. What do you observe? Check your oberservation for some polygon, *e.g.* a square.

4.4. (a) Let $\Delta_k := \operatorname{conv}(, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d)$ be the standard simplex in the lattice $\mathbb{Z}^d$. Prove that the number of lattice points in the *k-th* multiple of Δ_d for $k \in \mathbb{Z}_{\geq 1}$ is

$$\left| k\Delta_d \cap \mathbb{Z}^d \right| = \binom{d+k}{d}.$$

Hint: You may want to construct a bijection to the k-combinations with repetition from the set with $d+1$ elements.

(b) Show that

$$\sum_{k=0}^{\infty} \binom{d+k}{d} x^k = \frac{1}{(1-x)^{d+1}}.$$

Hint: You may want to use the bijection for the first part again to count ways to write x^n as a product of monomials.

4.5. Finish the exercises of Sheets 1, 2, and 3.