Integer Points in Polyhedra

Andreas Paffenholz — TU Berlin

Exercise Sheet 5 — Summer 19

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5.1. For integers p,q with $\gcd(p,q) = 1$ define the tetrahedron

$$\Delta_{pq} = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + p \cdot \mathbf{e}_2 + q \cdot \mathbf{e}_3)\,.$$

For $p = 1$ and $q = m$ the simplices are the *Reeve tetrahedra*

$$R_m := \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + m\mathbf{e}_3)$$

that we have already considered.

(a) Argue that its vertices are its only lattice points.

Remark: The Theorem of White tells us that also the converse is true: Every lattice tetrahedron with only four lattice points is unimodularly equivalent to a Δ_{pq}.

(b) Compute the Ehrhart polynomial and the $h^\star$-polynomial of Δ_{pq}.

(c) For which parameters are Δ_{pq} and $\Delta_{p'q'}$ unimodularly equivalent?

5.2. Let p be a polynomial of degree d such that

$$\sum_{k \geq 0} p(k) t^k \;=\; \frac{h_0^* \;+\; h_1^* t \;+\; h_2^* t^2 \;+\; \cdots \;+\; h_d^* t^d}{(1-t)^{d+1}}\,.$$

Show that

$$p(-1) \;=\; p(-2) \;=\; \cdots \;=\; p(d-r) \;=\; 0 \quad\text{and}\quad p(-(d-r)) \;\neq\; 0$$

if and only if

$$h_d^* \;=\; h_{d-1}^* \;=\; \cdots \;=\; h_{r+1}^* \;=\; 0 \quad\text{and}\quad h_r^* \;\neq\; 0\,.$$

5.3. For a d-dimensional lattice polytope let $r := \operatorname{codeg} P := d + 1 - \deg P = d + 1 - \deg h_P^*$ be the *codegree* of P. Show that

$$\min_k(|k \cdot \operatorname{int} P \cap \mathbb{Z}^d| \;\neq\; 0) \;=\; r$$

and

$$|r \cdot \operatorname{int} P \cap \mathbb{Z}^d| \;=\; h_r^*\,.$$

Hint: You may want to use Exercise 5.2.

5.4. Let P, Q be lattice polytopes with $Q \subseteq P$. Show that there exists a regular triangulation that restricts to a regular triangulation of Q.

Hint: You may want to show that you can extend a weight function for a triangulation of Q to a weight function for a triangulation of P.

5.5. Let $P, Q \subseteq \mathbb{R}^d$ be lattice polytopes with $\dim P = d$ and $Q \subseteq P$. Show that their h^*-polynomials h^*_P and h^*_Q satisfy

$$h^*_{Q,i} \leq h^*_{P,i}\,.$$

Hint: You may want to use Exercise 5.4.

5.6. Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope. A *lattice pyramid* $\operatorname{Pyr}(P)$ over P is $\operatorname{conv}(P \times \{0\}, (\mathbf{x}, 1))$ for any $\mathbf{x} \in \mathbb{Z}^d$.

Show that the h^*-polynomials of P and $\operatorname{Pyr}(P)$ coincide.

Hint: You may want to use Exercise 5.5

5.7. Finish the exercises of Sheets 1, 2, 3, and 4.