Integer Points in Polyhedra

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Exercise Sheet 6 Summer 19
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6.1. Prove that the coefficients of the Ehrhart polynomial of a d-dimensional lattice polytope are in $\mathbb{Z}/d!$.

6.2. Let P be a d-dimensional lattice polytope with Ehrhart polynomial $\sum_{k=0}^{d} c_k t^k$. Show that

$$c_{d-1} = \frac{1}{2}\operatorname{vol}(\partial P).$$

Here, $\operatorname{vol}(\partial P)$ denotes the surface area of P, namely,

$$\operatorname{vol}(\partial P) := \sum_{F \in \mathsf{F}(P)} \operatorname{vol}(F),$$

where $\mathsf{F}(P)$ is the set of facets of P and $\operatorname{vol}(F)$ denotes the (non-normalized) volume with respect to the lattice $\operatorname{aff}(F) \cap \mathbb{Z}^d$.

6.3. A simplex which is unimodularly equivalent to the standard simplex is called *unimodular*. A triangulation is *unimodular* if all its simplices are.

(a) For a k-dimensional unimodular simplex Δ and $t \in \mathbb{Z}_{\geq 1}$ show that

$$|\mathbb{Z}^k \cap \operatorname{relint}(t\Delta)| = \binom{t-1}{k}.$$

(b) Suppose P admits a unimodular triangulation $\mathscr{T}$ with $f_0(\mathscr{T})$ vertices, $f_1(\mathscr{T})$ edges, …, $f_d(\mathscr{T})$ d-simplices. Show that

$$e_P(t) = \sum_{k=0}^{d} f_k(\mathscr{T}) \binom{t-1}{k}.$$

(c) Conclude that any two unimodular triangulations have the same f-vector $(f_0, \dots, f_d)$.

6.4. Let $K \subseteq \mathbb{R}^d$ be a centrally symmetric convex body with $\operatorname{int}(K) \cap \mathbb{Z}^d = \{0\}$ and $\operatorname{vol}(K) = 2^d$. Show that K is a polytope with $2(2^d - 1)$ facets and each facet of K contains at least one lattice point in its relative interior.

Hint: To prove polytopality choose, for any non-zero lattice point $\mathbf{x}$, a half space $H_{\mathbf{x}}$ that has $\mathbf{x}$ not in its interior and contains K, let $S_{\mathbf{x}} := H_{\mathbf{x}} \cap -H_{\mathbf{x}}$, and consider the intersection of all $S_{\mathbf{x}}$.

For the number of facets you may want to consider midpoints between lattice points in different facets.

6.5. Let $K \subset \mathbb{R}^d$ be a centrally symmetric convex set with $\operatorname{int}(K) \cap \Lambda = \{0\}$. Then $|K \cap \Lambda| \leq 3^d$.

6.6. (a) Determine the set of all pairs $(h_1^\star, h_2^\star)$ of non-negative integers such that the polynomial $h^\star(t) := h_2^\star t^2 + h_1^\star t + 1$ is the $h^\star$-polynomial of a lattice polygon P. Draw your set.

(b) Deduce a corresponding clasification for the coefficients of the Ehrhart polynomial.

Hint: You may want to recall Scott's Theorem.

6.7. Finish the exercises of Sheets 1, 2, 3, 4, and 5.