Integer Points in Polyhedra

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Exercise Sheet 7 | Summer 19
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7.1. Compute the *integer point generating series*, *Ehrhart series*, *$h^\star$-polynomial* and the *Ehrhart polynomial* for

(a) the polytope $P \subseteq \mathbb{R}^2$ spanned by $\mathbf{e}_1, \mathbf{e}_2$ and $-\mathbf{e}_1 - \mathbf{e}_2$.

(b) the polytope $P \subseteq \mathbb{R}^3$ spanned by $\mathbf{e}_1, \mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

(c) the polytope $R := \operatorname{conv}(P \times \{0\}, \mathbf{e}_3, -\mathbf{e}_3) \subseteq \mathbb{R}^3$.

7.2. Prove

(a) $\mu(\mathbb{Z}^d) \ = \ \sqrt{d}/2$

(b) $\mu(D_3) \ = \ 1$

(c) $\mu(D_n) \ = \ \sqrt{n}/2$ for $n \geq 4$

7.3. (a) Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 1}$. Show that there is a positive integer q greater than N (alternatively: infinitely many q) and some integer p (alternatively: one for for each q) such that

$$\left| \alpha - \frac{p}{q} \right| \ \leq \ \frac{1}{q^2}\,.$$

Hint: For some $M \in \mathbb{Z}_{\geq 1}$ you may want to consider the set

$$S \ := \ \{(x, y) \in \mathbb{R}^2 \mid |\alpha x - y| \leq 1/M, |x| \leq M\}\,.$$

(b) Given $\alpha_1, \dots, \alpha_n \in \mathbb{R}$and $M \in \mathbb{Z}_{\geq 1}$ show that there a positive integer $q > M$ and integers $p_1, \dots, p_n$ such that for $1 \leq i \leq n$

$$\left| \alpha_i - \frac{p_i}{q} \right| \ \leq \ \frac{1}{q^{1+1/n}}\,.$$

7.4. Let $\Lambda_0 \subseteq \Lambda \subseteq \mathbb{R}^d$ be lattices. Show that

$$\rho(\Lambda) \ \leq \ \rho(\Lambda_0) \ \leq \ \left|\Lambda/\Lambda_0\right| \rho(\Lambda)\,.$$

7.5. Let Λ be a lattice in $\mathbb{R}^d$ with dual lattice $\Lambda^\star$. Then

$$4\mu(\Lambda) \cdot \rho(\Lambda^\star) \ \geq \ 1$$

7.6. Finish the exercises of Sheets 1, 2, 3, 4, 5, and 6.