Integer Points in Polyhedra

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Exercise Sheet 8 | Summer 19

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8.1. Let $C = \operatorname{cone}(\mathbf{a}_1, \dots, \mathbf{a}_k) \subseteq \mathbb{R}^d$ be a cone of dimension $e \le d$. The *polar* cone is $C^\vee := \{\alpha \mid \alpha(\mathbf{x}) \le 0 \text{ for all } \mathbf{x} \in C\}$.

(a) Show that $(C^\vee)^\vee = C$.

(b) Show that $C^\vee$ contains a linear space if $e < d$.

(c) Prove that the polar of a unimodular cone is a unimodular cone.

8.2. Let $C = \operatorname{cone}(\mathbf{a}_1, \dots, \mathbf{a}_d) \subseteq \mathbb{R}^d$ be a cone of dimension d and let $D_1, \dots, D_k$ be a signed (or some other) decomposition of the cone $C^\vee$, so that, for some $\varepsilon_i \in \{1, -1\}$, $1 \le i \le k$ we have

$$G_{C^\vee}(\mathbf{x}) = \sum_{i=1}^{d} \varepsilon_i G_{D_i}(\mathbf{x}) + \text{contributions from lower dimensional cones}.$$

(a) Show that

$$G_C(\mathbf{x}) = \sum_{i=1}^{d} \varepsilon_i G_{D_i^\vee}(\mathbf{x}) + \text{contributions from cones with lineality}.$$

and deduce that

$$g_C(\mathbf{x}) = \sum_{i=1}^{d} \varepsilon_i g_{D_i^\vee}(\mathbf{x}).$$

(b) Test this equation on some cones of your choice.

8.3. Compute the Ehrhart or h^*-polynomial using Barvinok's algorithm of

(a) $P = [1, 5]$

(b) $P = [0, 1]^2$

(c) some other polytope of your choice.

You can use a primal decomposition as in the lecture, or the dual decomposition of Exercise 8.2. For the evaluation you may want to use a computer.

8.4. Prove

(a) $\rho(\mathbb{Z}^d) = \frac{1}{2}$

(b) $\rho(A_n) = \rho(D_n) = \frac{\sqrt{2}}{2}$ for $n \ge 2$

(c) $\rho(D_2^+) = \frac{1}{2\sqrt{2}}$

(a) $\rho(D_4^+) = \frac{1}{2}$

(b) $\rho(D_6^+) = \sqrt{\frac{3}{8}}$

(c) $\rho(D_n^+) = \frac{\sqrt{2}}{2}$

8.5. Finish the exercises of Sheets 1, 2, 3, 4, 5, 6, and 7.