Integer Points in Polyhedra

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Exercise Sheet 9
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9.1. Let $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ be lattice polytopes. The *product* $P \times Q$ of P and Q is

$$P \times Q \ := \ \left\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in P,\ \mathbf{y} \in Q \right\} ,$$

and the join $P \star Q$ is

$$P \star Q \ := \ \operatorname{conv}\left(P \times \{\mathbf{0}\} \times \{0\},\ \{\mathbf{0}\} \times Q \times \{1\} \right) .$$

(a) Show that the Ehrhart polynomial of $P \times Q$ is the product of the Ehrhart polynomials of P and Q.

(b) Show that the h^*-polynomial of $P \star Q$ is the product of the h^*-polynomials of P and Q.

9.2. Let P be a polytope that contains $\mathbf{0}$ in its interior. The dual polytope is

$$P^\vee \ := \ \{\mathbf{a} \in (\mathbb{R}^d)^* \mid \mathbf{a}(\mathbf{x}) \ \geq -1 \ \forall\, \mathbf{x} \in P\} .$$

Prove the following:

(a) It suffices to check $\mathbf{a}(\mathbf{x}) \geq -1$ for all vertices of P.

(b) The vertices of P are facet normals of the dual.

(c) We have a bijective correspondence between facets P and vertices of $P^\vee$.

(d) $(P^\vee)^\vee \ = \ P$.

9.3. A lattice polytope $P := \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is *reflexive* if there is a lattice point $\mathbf{w} \in P$ such that all facets have lattice distance 1 from $\mathbf{w}$. Prove the following:

(a) $\mathbf{w}$ is the unique interior point of P.

(b) If all rows of A are primitive and integral, then $\mathbf{b} = \mathbf{1}$.

(c) If $\mathbf{w}$ is the origin, then $P^\vee$ is a lattice polytope.

(d) If $\mathbf{w}$ is the origin, then $P^\vee$ is reflexive.

9.4. Let P be a reflexive polytope. Show that the number of interior points in $2P$ equals the number of lattice points in P.

Determine the Ehrhart and h^*-polynomial of a 3-dimensional reflexive lattice polytope.

9.5. Show that a polygon $P \subseteq \mathbb{R}^2$ is reflexive if and only if it has a unique interior lattice point.

Show that this is not true in higher dimensions.

9.6. Finish the exercises of Sheets 1 to 8.