Integer Points in Polyhedra

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Exercise Sheet 10
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10.1. Let P be a reflexive polytope in a lattice Λ.

For $\mathbf{x}, \mathbf{y} \in \partial P \cap \Lambda,\ x \neq y$ we write $\mathbf{x} \sim \mathbf{y}$ if there exists a facet of P containing $\mathbf{x}$ and $\mathbf{y}$.

For distinct $\mathbf{x}, \mathbf{y} \in \partial P \cap \Lambda$

(a) either $x \sim y$
(b) or $x + y = 0$
(c) or $x + y \in \partial P \cap \Lambda$.

If (c) holds, then

- $\mathbf{x} \sim \mathbf{x} + \mathbf{y}$ or $\mathbf{y} \sim \mathbf{x} + \mathbf{y}$ and
- there are $a, b \in \mathbb{Z}_{\geq 1}$ such that for $\mathbf{z} := a\mathbf{x} + b\mathbf{y} \in \partial P \cap \Lambda$ we have $\mathbf{x} \sim \mathbf{z} \sim \mathbf{y}$ and $a = 1$ or $b = 1$.

Conclude that the diameter of the graph of a simplicial reflexive polytope is at most 3.

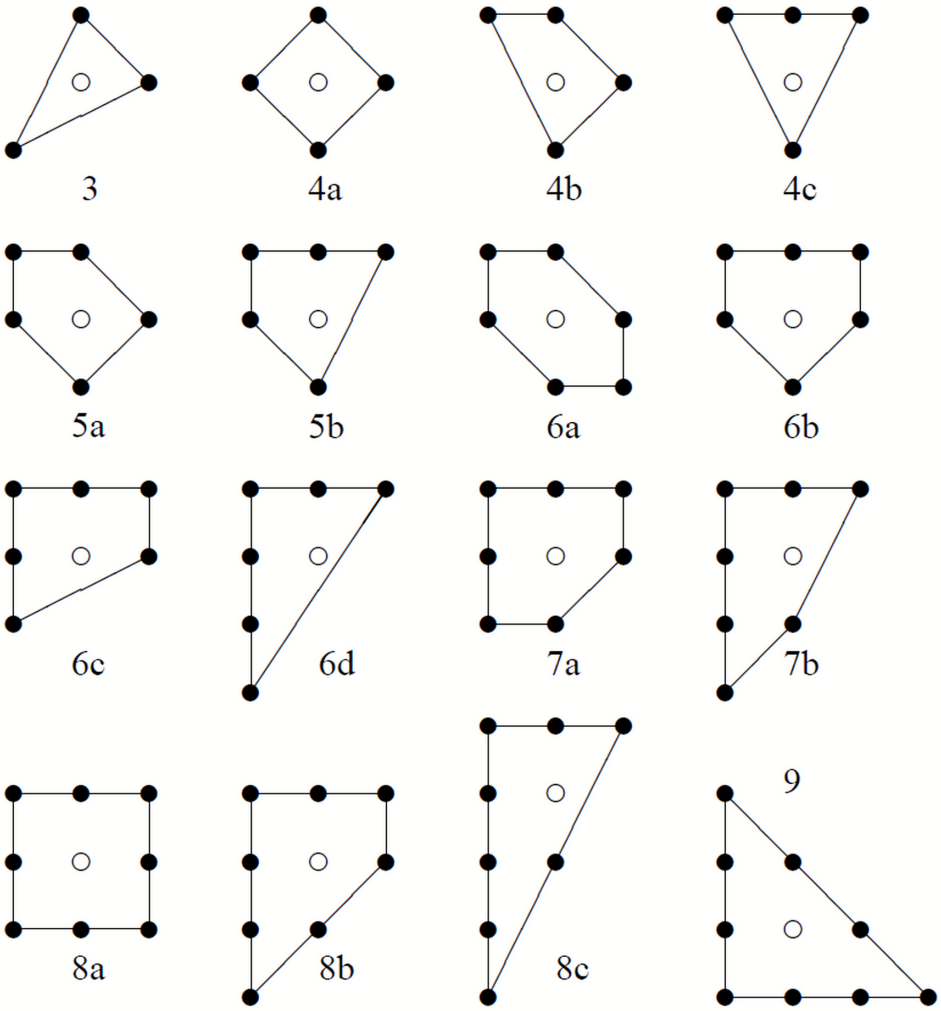
10.2. Prove the classification of reflexive polygons given on the back.

10.3. Show that a polytope P is Gorenstein of index r if and only if for all $k \geq r$

$$e_P(-k) \;=\; (-1)^d e_P(k-r)\,.$$

10.4. Prove that the Birkhoff polytope B_n (the convex hull of $n \times n$-permutation matrices) is a Gorenstein polytope of codegree n. What is its dimension and degree? What is the unique interior lattice point of nB_n ?

10.5. Finish the exercises of Sheets 1 to 9.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