# Integer Points in Polyhedra

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Exercise Sheet 11
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11.1. Let $P$ be a Gorenstein polytope of index $r$, $\mathbf{w}$ the unique inner point of $rP \cap \Lambda$ and

$$P^{\times} \ := \ \mathbf{x} \in C_P^{\vee} \ \mid \ \langle \mathbf{x}, \mathbf{w} \rangle \ = \ 1\} \, ,$$

where $C_P^{\vee}$ is the cone dual to the cone over $P$.

(1) Show that $P^{\times}$ is a Gorenstein polytope of index $r$.
(2a) Show that for $r = 1$ and $\mathbf{w} = \mathbf{0}$ this coincides with $P^{\vee}$.
(2b) Show that this is not true for $r > 1$.

11.2. Show that any pair of vertices of a Gorenstein polytope of index $r > 1$ either is in a common face or is an antipodal pair (with respect to the point $\mathbf{w}$, where $\mathbf{w}$ is the unique interior point of $rP$).

11.3. A lattice polytope is *integrally closed* or has the *IDP property* if each lattice point in $kP$ is the sum of $k$ lattice points in $P$, for every $k \geq 1$.

Show that a lattice polytope is integrally closed if it admits a unimodular triangulation.

* 11.4. If $P$ has a (regular) unimodular triangulation $\mathscr{T}$ then its dilation $cP$ has one too, for every positive integer $c$.

Hint: Use *lattice dicing* to subdivide multiples of simplices. Convince yourself that you can do this in a way that the subdivisions on the boundary coincide. Now turn this into a triangulation.

11.5. Let

$$A_{n-1} := \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i, j \leq n\}$$

be the root system of type $A$. Show via lattice dicing that lattice polytopes with facet normals in $A_{n-1}$ have a regular unimodular triangulation.

*** 11.6. Let

$$B_n := \{\pm \mathbf{e}_i \mid 1 \leq i \leq n\} \cup \{\pm \mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i < j \leq n\}$$

be the root system of type $B$. $B_n$ splits into the sets of *long* roots $\pm\mathbf{e}_i \pm \mathbf{e}_j$ and *short* roots $\pm\mathbf{e}_i$. Show that you can use a “partial” lattice dicing using only the short roots to prove that that lattice polytopes with facet normals in $B_n$ have a regular unimodular triangulation.

11.7. Finish the exercises of Sheets 1 to 10.