Integer Points in Polyhedra

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Some Solutions for Exercise Sheet 3 — Summer 19

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∘ 3.2. Show that any lattice polytope P has a regular triangulation using only the vertices of P.

Solution: By passing to the affine hull of P if necessary we can assume that P is full dimensional. Let $V := \mathsf{V}(P) = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be the m vertices of the polytope. We want to show that any sufficiently generic vector $\mathbf{w}$ induces a regular triangulation.
Let any height vector $\mathbf{w} = (w_i)_{1 \le i \le m}$ be given and let

$$Q \ := \ \operatorname{conv}\left((w_i, \mathbf{v}_i) \mid 1 \le i \le m\right)$$

be the lift of P with respect to $\mathbf{w}$. As P is convex, all $\mathbf{v}_i$ are necessarily part of the subdivision induced by the projection of the lower hull of Q. Hence, we only need to deal with the question whether the subdivision is a triangulation.
Now the subdivision induced by $\mathbf{w}$ is a triangulation if and only if each facet of the lower hull of Q is a d-simplex, *i.e.* if at most $d+1$ of the points

$$(w_1, \mathbf{v}_1), \dots, (w_m, \mathbf{v}_m)$$

lie on a common hyperplane. We can derive equations that charaterize those $\mathbf{w}$ where some $d+2$ of the points lie on a common hyperplane. Namely, for any $(d+2)$-tuple

$$(w_{i_1}, \mathbf{v}_{i_1}), \dots, (w_{i_{d+2}}, \mathbf{v}_{i_{d+2}})$$

being on a common hyperplane means that the determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ w_{i_1} & w_{i_2} & \cdots & w_{i_{d+2}} \\ \mathbf{v}_{i_1} & \mathbf{v}_{i_2} & \cdots & \mathbf{v}_{i_{d+2}} \end{pmatrix}$$

vanishes (and the linear relation among the columns that then must exist allows to represent each column as an affine combination of the others). We can view this determinant as a linear functional in the entries of $\mathbf{w}$. Such a functional defines a hyperplane in the space of all $\mathbf{w}$. Each such hyperplane is a zero set in this space. Hence, its complement is non-empty.
There are $\binom{m}{d+2}$ different such functionals defining a hyperplane. As this is a finite number, also the complement of all these hyperplanes is non-empty. Choosing any $\mathbf{w}$ in this complement satisfies our requirements.