Integer Points in Polyhedra

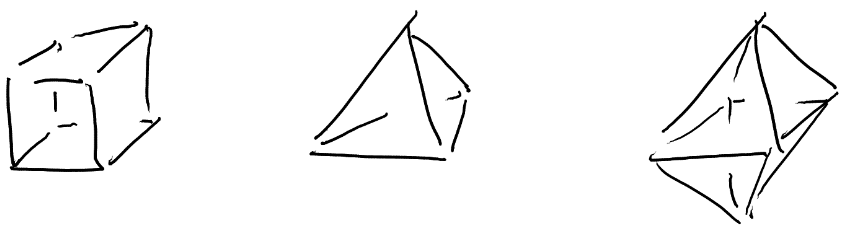
0. What

lattice : $\mathbb{Z}^d$

polytope : $\mathrm{conv}(X)$ for $X \subseteq \mathbb{R}^d$ finite

lattice polytope P : $\mathrm{conv}(X)$ for $X \subseteq \mathbb{Z}^d$ finite

examples
- cube $[0,1]^d$
- simplex $\mathrm{conv}(0, e_1, \dots, e_d)$
- cross polytope $\mathrm{conv}(\pm e_1, \dots, \pm e_d)$



Some key questions:

- enumerate or count lattice pts (in its interior or on the boundary)
- does it have interior lattice points
- does $X \subseteq \mathbb{Z}^d$ impose restrictions on the geometry?
- what are useful invariants?

- can we classify / characterize (the structure of) lattice polytopes (with additional properties, e.g. few lattice points, no lattice points in interior)

Why should we study lattice polytopes?

→ at intersection of several fields:

geometry, algebra, number theory

(statistics, optimization, alg. geometry)

→ basic structure found in many applications

- convex geometry / geometry of numbers

use geometric methods to study algebraic integers, solve Diophantine equations

Minkowski's Thm: K convex, centrally symmetric, large

$\Rightarrow$ K contains non-zero lattice pt

- Ehrhart Theory: counting / enumerating lattice pts via generating functions

Thm of Ehrhart: $|kP \cap \mathbb{Z}^d|$ given by polynomial in k

- algorithms and integer linear programming:
    - efficient counting / enumeration
    - short lattice vectors (lattice bases, flatness, improving direction)
    - integer linear programming in fixed dimension
- geometric combinatorics
    - polyhedra, simplicial complexes
    - counting faces, incidences
- enumerative combinatorics
    - counting combinatorial objects
      → Frobenius problem
      → latin squares
- (combinatorial) commutative algebra

  lattice pt $u \in \mathbb{Z}^d \leftrightarrow$ monomial x^u

  $A := \{a_1, \dots, a_d\} = P \cap \mathbb{Z}^d \subseteq \mathbb{Z}^d \setminus \{0\}$

  $\pi : \mathbb{N}^n \to \mathbb{Z}^n, \; u \mapsto \sum a_i u_i$

  $$\hat{\pi} : \Bbbk[x] = \Bbbk[x_1, \dots, x_n] \longrightarrow \Bbbk[t^{\pm 1}] = \Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$$
  $$x_j \longmapsto t^{a_j}$$

  toric ideal $\hat{=}$ kernel of $\hat{\pi}$

- toric geometry ⊆ algebraic geometry
  → toric varieties, roots of polynomials
  theoretical physics

clearly we can't touch on all of these topics

focus on
- lattice polytopes and Ehrhart Theory
- geometry of numbers
- algorithms for counting / enumeration
  integer linear programming
- structure and classifications
- triangulations (/ toric geometry)

Plan:
- low dimensional lattice polytopes
- lattices and linear algebra
  lattice bases, equivalence, normal forms
- generating functions
- counting and Ehrhart Theory
- Reciprocity
- Geometry of numbers
  Minkowski / LLL / flatness / ILP
- Efficient counting
  Barvinok

1.5

- Classifications and Structure
  - few lattice points
  - few — " — in interior → finiteness
  - no lattice pts in interior

- Duality
  reflexive / Gorenstein polytopes

- Triangulations
  regular / unimodular
  existence
  toric ideals
  Fano polytopes

Technical Stuff:

• Class:	Wed	10-12	744
	Thu	10-12	621
• Tutorial	Thu	14-16	621

Books: Barvinok, A course on convexity
Beck, Robins: Counting the cont. discretely
DeLoera, Hemmecke, Köppe:
Algebraic and Geometric Ideas in the Theory of Discrete Opt

1.6

Bertsimas, Weismantel: Opt. over Integers

Exams: Oral

Tutorials: Sheets around Tuesday, Discuss on Thursday

Question: How much convex geometry?

Office: TUT 622, just come around if you have questions

1. Low Dimensional Lattice Polytopes

Def: lattice polytope P : $P = \mathrm{conv}(X)$ for $X \subseteq \mathbb{Z}^d$ finite

let us consider low dimensional lattice polytopes:

- $d=0$ $\rightarrow$ just one polytope $P = \{0\} \in \mathbb{Z}^0$
- $d=1$ $\rightarrow$ intervals $P = [a,b]$ for $a,b \in \mathbb{Z}$

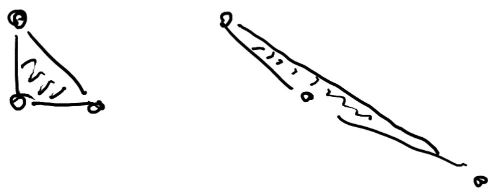
  length: $b-a$

  # lattice points : $b-a+1$

  for $t \in \mathbb{Z}$: $[a,b]$ structurally equivalent to $[a+t, b+t]$

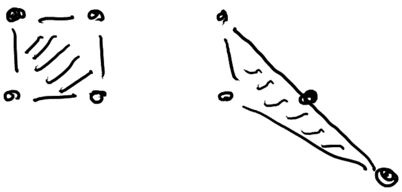
  $\leadsto$ only invariant is length or #lp

- $d=2$:

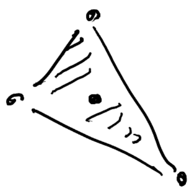


3 lp
vol 1/2

we can study volume
no of (interior | boundary) lattice pts



4 lp
vol 1



4 lp
vol 3/2

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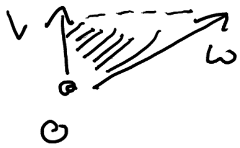
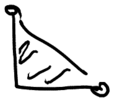
How do we compute volume:

triangle Δ



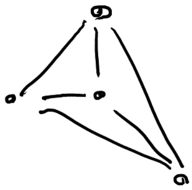
v, w are basis $\to$ basis change matrix T

$Tv = e_1$, $Tw = e_2$

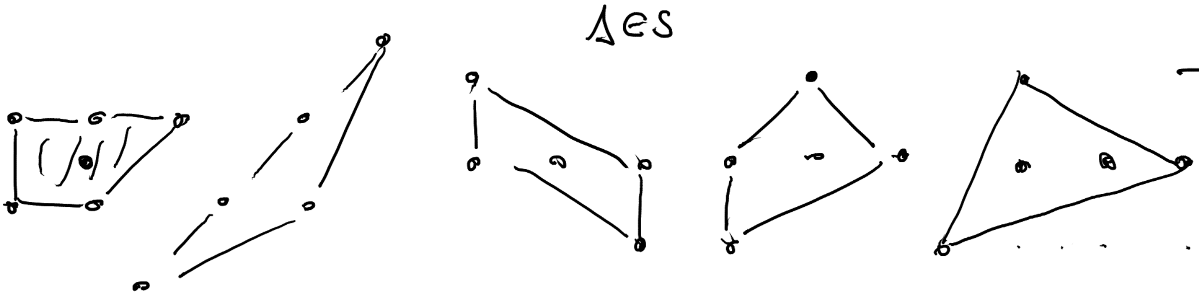
$T\Delta =$  with vol $1/2$

$\Rightarrow$ vol $\Delta = \frac{1}{2} \det T$

general polygons: subdivide into triangles



Triangulation of P:
collection S of triangles with
- $\Delta \neq \Delta' \in S \Rightarrow \Delta \cap \Delta'$ is empty, a vertex or an edge of both
- $\bigcup_{\Delta \in S} \Delta = P$



5lp vol $3/2$ 5lp vol 2 5lp vol $5/2$

guess formula for volume from lattice pts

3lp $\Rightarrow$ vol $1/2$, $a = i + \frac{b}{2} - 1$