2.1

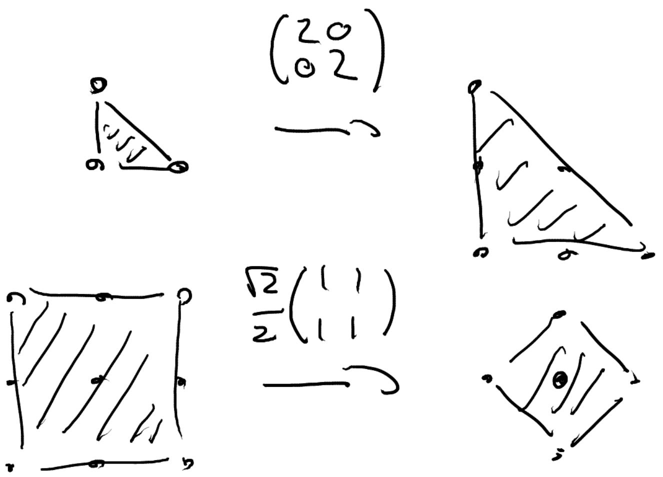
→ want to classify polygons according to this

But first need to clarify equivalence!

Recall from linear geometry:

$\varphi: \mathbb{R}^d \to \mathbb{R}^d$ is an automorphism if φ is linear and bijective

equivalently: $\varphi: x \mapsto Ax + t$ for $A \in GL_d(\mathbb{R})$



does neither preserve #lp nor volume

(and restricting to $O_d(\mathbb{R})$ still does not preserve #lp)

→ linear (orthonormal) automorphisms are not useful.

instead: consider maps

$\varphi: \mathbb{Z}^d \to \mathbb{Z}^d$ linear and bijective equiv: $\varphi : x \mapsto Ax + t$ for

$A \in GL_d(\mathbb{Z}) = \{A \in GL_d(\mathbb{R}) \mid A, A^{-1} \in \mathbb{Z}^{d\times d}\}$

$\varphi : x \mapsto Ax + t$, $A \in GL_d(\mathbb{Z})$, $t \in \mathbb{Z}^d$

is an *affine unimodular transformation*

Prop / Ex: $|\det A| = 1$

$A \in GL_d(\mathbb{Z})$ iff $Ax \in \mathbb{Z}^d \Leftrightarrow x \in \mathbb{Z}^d \quad \forall x \in \mathbb{Z}^d$

Def P, Q lattice polytopes

Then P and Q are isomorphic (unimodularly equivalent) if

$Q = \varphi(P)$ for an affine unimodular transformation

Note / Ex $\varphi, \varphi^{-1} : \mathbb{Z}^d \to \mathbb{Z}^d$ are bijections

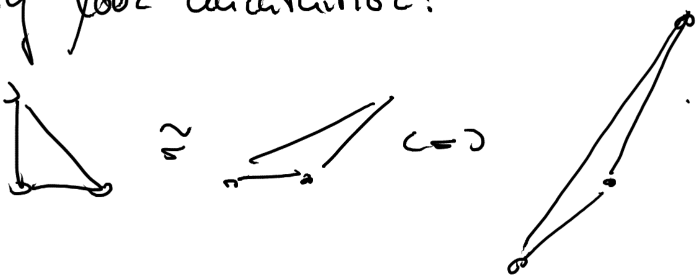
$\varphi^{-1} : x \mapsto A^{-1}(x - t)$

$$P \cap \mathbb{Z}^d \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} Q \cap \mathbb{Z}^d$$

Prop: Isomorphic lattice polytopes have the same number of (interior / boundary) lattice pts and the same volume

proof: for the volume note $|\det A| = 1$

may look counterintuitive:



we mostly consider lattice polytopes up to unimodular equiv.

→ angles and Euclidean distances are <u>not</u> preserved by unimodular transformations

<u>Def</u> lattice length of segment $e = \mathrm{conv}(p,q)$: $\mathrm{length}(e) = |e \cap \mathbb{Z}^d| - 1$

→ <u>preserved!</u>

<u>Pick's Thm</u>

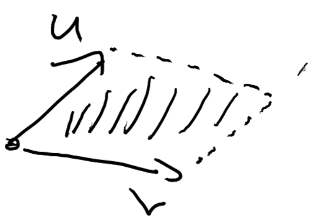
P lattice polygon $|P \cap \mathbb{Z}^2| = 3$

Then $\mathrm{vol}\, P = 1/2$ (and $P \cong \mathrm{conv}(0, e_1, e_2)$)

<u>proof</u>: we can assume that $P = \mathrm{conv}(0, u, v)$

consider $\Pi := \{ \lambda u + \mu v \mid 0 \le \lambda, \mu < 1 \}$

half-open parallelepiped



$u, v \notin \Pi$!

$\mathrm{vol}\, \Pi = |\det(u,v)| = 2\, \mathrm{vol}\, P$

2.4

<u>claim</u> $\Pi \cap \mathbb{Z}^2 = \{0\}$

Suppose $\omega := \lambda u + \mu v \in \Pi \cap \mathbb{Z}^2$ with $0 \le \lambda, \mu < 1$

Two cases:

$\lambda + \mu \le 1 \quad \Rightarrow \omega \in P \Rightarrow \omega = 0$

$1 < \lambda + \mu < 2$

Then $(u+v) - \omega = (1-\lambda)u + (1-\mu)v$

and $0 < (1-\lambda) + (1-\mu) = 2 - (\lambda + \mu) < 1$

and $(u+v) - \omega \in P \setminus \{0\}$, a contradiction

Hence: Any lattice pt in $\mathbb{Z}^2$ can be written as an integer linear combination of u, v:

$$\mathbb{Z}^2 \ni \omega = \underset{\in \mathbb{R}}{\lambda} u + \underset{\in \mathbb{R}}{\mu} v = \lfloor \lambda \rfloor u + \lfloor \mu \rfloor v + \underbrace{\{\lambda\} u + \{\mu\} v}_{\in \Pi, \text{ so } = 0}$$

In particular:

$$e_1 = \lambda_1 u + \mu_1 v \qquad \text{for } \lambda_i, \mu_i \in \mathbb{Z}$$
$$e_2 = \lambda_2 u + \mu_2 v$$

i.e. $$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \det(u,v), \dfrac{1}{\det(u,v)} \in \mathbb{Z} \Rightarrow |\det(u,v)| = 1$

$\Rightarrow \operatorname{vol} P = \frac{1}{2}$ and $P \cong \operatorname{conv}(0, e_1, e_2)$

$\boxed{\Sigma}$

Actually, we proved more:

- $\Pi + (\lambda u + \mu v)$ for $\lambda, \mu \in \mathbb{Z}$ tile $\mathbb{R}^2$
- any $x \in \mathbb{R}^2$ is a sum of an integer linear combination of u, v and a pt of Π
- if $\Pi \cap \mathbb{Z}^2 = \{0\}$, then u, v generate $\mathbb{Z}^2$

<u>Def</u> A basis u, v of $\mathbb{R}^2$ as a vector space is a lattice basis of $\mathbb{Z}^2$ if

$$\mathbb{Z}^2 = \{ \lambda u + \mu v \mid \lambda, \mu \in \mathbb{Z} \}$$

Note/Ex: Pick's Thm is false in higher dimensions:

$$R(m) := \operatorname{conv}(0, e_1, e_2, e_1 + e_2 + m e_3)$$

Then $$|R_m \cap \mathbb{Z}^3| = 4$$

but $$\operatorname{vol} R_m = \frac{1}{3} \frac{1}{2} m = m/6$$

so $R_m \not\cong R_{m'}$ for $m \neq m'$!

We have established that there is, up to isomorphism, exactly one lattice triangle of minimal volume $1/2$

2.6

Pick's Formula:

P lattice polygon, i interior lattice pts

b boundary —— " ——

a volume

Then $$a = i + b/2 - 1$$

proof: Ex ☒

what about dilates of polygons

$$kP := \{kx \mid x \in P\}$$

clearly $$a(k) := vol(kP) = k^2 a$$
$$b(k) = kb$$

$$\Rightarrow \quad i(k) = a(k) - \frac{b(k)}{2} + 1$$
$$= ak^2 - \frac{b}{2}k + 1,$$ a poly of deg 2!

maybe more natural to look at all lattice pts:

$$\ell(k) := i(k) + b(k) = a(k) + \frac{b(k)}{2} + 1$$

$$= ak^2 + \frac{b}{2}k + 1,$$ a poly of deg 2!

$$= i(-k)$$ reciprocity

$\rightarrow$ Ehrhart's Theorem

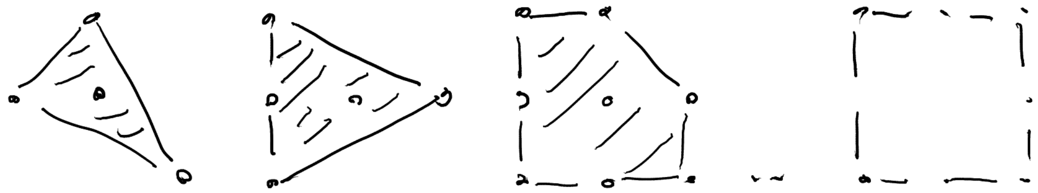
2.7

Are there restrictions on i, b or a?

$i = 0$

b, a can be arbitrarily large

$i = 1$



$a = 3/2$ $\quad$ $a = 2$ $\quad$ $a = 5/2$ $\quad$ $a = 4$

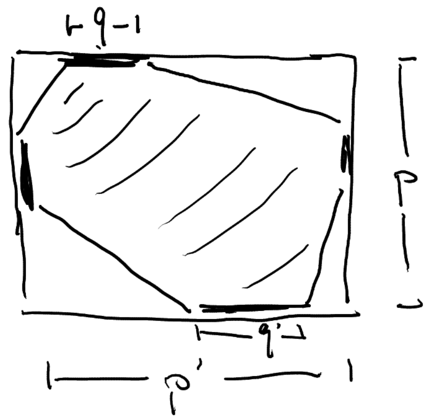
Can we get $a > 4$ for $i = 1$?

<u>Scott's Thm</u> P lattice polytope with $i \geq 1$ interior lattice pts

Then either

- $P \cong 3\Delta_2$, so $a = 9/2$, $i = 1$, $b = 9$ or
- $a \leq 2(i+1) \iff b \leq a + 4$

proof place P into rectangle $R = [0, p'] \times [0, p]$ with p as small as possible (mod. lattice transformations)

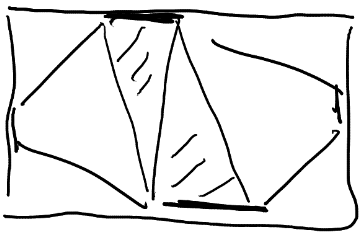


Then $2 \leq p \leq p'$

Let q, q' be the intersection with top / bottom edge.

At most $2(p-1)$ boundary lattice pts
are not on q, q'

$\Rightarrow b \leq (q+1)+(q'+1)+2(p-1) = q+q'+2p$ (*)



$a \geq \frac{b}{2}$ (Pick) (**)

$a \geq \frac{1}{2}p(q+q')$ (***)

(*), (***)

$2b-2a \leq 2(q+q'+2p)-p(q+q') = (q+q'-4)(2-p)+8$ (+)

with strict ineq. if P has a vertex not on q, q'

Case distinction

(I) $p = q+q' = 3$: use (+), gives special case

(II) $p = 2$ or $q+q' \geq 4$: use (+)

(III) $p = 3$ and $q+q' \leq 2$: use (+), (**)

(IV) $p \geq 4$ and $q+q' \leq 3$: requires some more work.