Similarly we can use row transformations:

Thm (<u>Smith Normal Form</u> SNF)

$A \in \mathbb{Z}^{d \times n}$. Then there are unimod. transformations L, R s.th.

$$S := L A R$$

is a diagonal matrix

$$S = \begin{pmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_d \end{pmatrix}$$

with $s_1 \mid s_2 \mid s_3 \mid \dots \mid s_d$ and $s_i \in \mathbb{Z}_{\geq 0}$

The matrix S is unique (but L, R are not!)

For the proof use both row and column transformations $\boxtimes$

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thm (Basis of sublattice)

Λ' sublattice of Λ, $\operatorname{rk} \Lambda = d$

Then there is a lattice basis $u_1, \dots, u_d \in \Lambda$ of Λ and $s_1, \dots, s_d \in \mathbb{Z}_{\geq 0}$, $s_1 | s_2 | \dots | s_d$ s.th.

$$s_1 u_1, s_2 u_2, \dots, s_k u_k$$

is a basis of Λ', where $k := \max(i \mid s_i \neq 0)$

Let $\Lambda' \subseteq \Lambda$ be sublattice. For $x \in \Lambda$ we let

$$x + \Lambda' := \{x + y \mid y \in \Lambda'\}$$

be the coset of Λ modulo Λ'.
The set of cosets is Λ / Λ' and the index of Λ' in Λ is $\operatorname{index}_\Lambda \Lambda' := |\Lambda / \Lambda'|$

Def: Let $\Lambda = \mathbb{R}^d$ be a lattice. A <u>lattice polytope</u> is
is the convex hull of finitely many lattice points

<u>Remark</u>: for polytopes with vertices in $\mathbb{R}^d$ we can equally define
a polytope as the bounded intersection of a finite
number of half-spaces: $P = \{x \mid Ax \le b\}$

Deciding whether a polytope defined by
inequalities is a lattice polytope is a
computationally hard problem
($\rightarrow$ essentially we need a convex hull computation)

Examples:

(1) simplex, cube, cross polytope

hypersimplex: $C_d \cap \{x \mid \sum x_i = k\}$

0/1-polytopes / order polytopes

Birkhoff polytope

$$B_n = \{ A \in \mathbb{R}^{d \times d} \mid \sum_{i=1}^{d} a_{ij} = \sum_{i=1}^{d} a_{ji} = 1 \}$$

As in the 2D-case we should study equivalences on lattice polytopes.

Let $P \subseteq \mathbb{R}^d$ be a lattice polytope in a lattice Λ.

$\rightsquigarrow$ we can assume $\dim P = \operatorname{rank} \Lambda$

(otherwise pass to the space $\operatorname{aff} P$ with lattice $\operatorname{aff} P \cap \Lambda$)

Def Two lattice polytopes $P, P' \subseteq \mathbb{R}^d$ wrt to lattices Λ, Λ' are isomorphic / unimodularly equivalent if there is a lattice isomorphism $\varphi : \Lambda \to \Lambda'$ that maps the vertices of P to the vertices of P'

3. Triangulations

Recall: Polyhedral Complex C

Finite family of polyhedra (cells) s.th. $\forall P, Q \in C$:

(1) all faces of P are in C
(2) $P \cap Q$ is a face of both (maybe $\emptyset$)

P maximal $:\Leftrightarrow$ not properly contained in other cell
C pure $:\Leftrightarrow$ all max. cells have the same dimension
F facet in pure complex $:\Leftrightarrow$ maximal cell.

$S \subseteq C$ subcomplex $\Leftrightarrow$ S complex and cells in C.

Ex:
- polytopes
- boundary complex of a polytope.
- fans | normal fans

Def A subdivision of a polytope P is a pure polyhedral complex S s.th.
$$P = \bigcup_{c \in S} c$$

S is a triangulation if all its cells are simplices

S is "without new vertices" if $V(S) = V(P)$
"full" if $V(S) = P \cap \Lambda$

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Def: S subdivision of P

S is <u>regular</u> if there is a weight vector

$$\omega = (\omega_v)_{v \in V(S)} \in \mathbb{R}^{|V(S)|}$$

s.th. S is the <u>lower hull</u> of

$$L := \mathrm{conv}\left((v, \omega_v) \mid v \in V(S) \right)$$

where the lower hull is the polyhedral complex of the projection of those facets of L whose normal has negative last coordinate.

Given a set V of points and a weight vector $\omega \in \mathbb{R}^{|V|}$ we set

$$S_\omega := \text{lower hull of } \mathrm{lift}(V, \omega) := \mathrm{conv}((v, \omega_v))$$

<u>Note</u> (1) $V(S_\omega) \subseteq V$ but can be strict.

(2) may not be a triangulation.

(3) not all triangulations are regular.