6.1

Thm: Any (lattice) polytope P has a regular triangulation T with $V(T) = V(\Xi)$

proof. Ex $\square$

Cor: Any pointed cone C has a regular triangulation using only the rays of C.

proof: C pointed $\Rightarrow$ there is a functional α s.th. $\alpha(x) > 0 \ \forall x \in C$

$\Rightarrow P := C \cap \{x \mid \alpha(x) = 1\}$ polytope

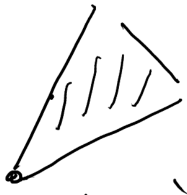
Take cones over triangulation of P. $\boxtimes$

4. Volume

Euclidean does not respect lattice:

Pick : vol of min. simplex is $1/2$ for $\Lambda = \mathbb{Z}^2$

$$\Lambda = \Lambda\left(\binom{1}{2}, \binom{?}{1}\right)$$



minimal, but vol = 1

$\rightsquigarrow$ volume relative to lattice:

$$\mathrm{vol}_\Lambda(\Pi(\Lambda)) := 1$$

Let P be a lattice simplex in Λ

Then
$$\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d \quad \text{with } \varphi(\Delta_d) = P$$
has integral matrix, so

$$d!\,\mathrm{vol}_\Lambda P = d!\,|\det\varphi|\,\mathrm{vol}_\Lambda \Delta_d = |\det\varphi|$$

so
$$d!\,\mathrm{vol}_\Lambda P \in \mathbb{Z}_{\geq 1}$$

and
$$d!\,\mathrm{vol}\,P = 1 \iff P \cong \Delta_d$$

Using a triangulation:

$d!\,\mathrm{vol}\,P \in \mathbb{Z}_{\geq 1}$ for all lattice polytopes.

<u>Def</u> The <u>normalized volume</u> of a d-dim lattice polytope P in $\mathbb{R}^d$ is

$$\operatorname{nvol} P := d!\, \operatorname{vol}_\Lambda P \in \mathbb{Z}_{\geq 1}$$

$\rightsquigarrow$ extend to lower dim by passing to $\operatorname{aff} P$ and $\Lambda \cap \operatorname{aff} P$.

P lattice polytope of nvol V
$\Rightarrow$ any triangulation has at most V maximal simplices!

$\rightsquigarrow$ show that this leaves only finitely many types

Recall: $S = \operatorname{conv}(v_0, \ldots, v_d)$ simplex, then $c(S) = \frac{1}{d+1} \sum v_i$ centroid of S.

<u>Thm</u> P d-dim lattice polytope, nvol $P = V$
Then there is $Q \cong P$ s.th. $Q \subseteq [0, dV]^d$
If P is a simplex, then even $Q \subseteq [0, V]^d$

<u>proof</u>: (1) $P = \operatorname{conv}(0, v_1, \ldots, v_d)$ simplex

$$M = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_d \\ | & & | \end{pmatrix}$$

6.4

HNF $\Rightarrow$ there is unimodular U s.th.

$U\Pi = H$ is
- upper triangular
- non-negative
- max elem of each column on diagonal.

$$\Rightarrow Q = \operatorname{conv}(0, h_1, \dots, h_d) \subseteq [0, \underbrace{\textstyle\prod h_{ii}}_{=V}]^d$$

(2) in general:

In the exercises you prove that there is a simplex $S \subseteq P$ s.th.

$$P \subseteq (-d)(S - c(S)) + c(S)$$
$$= (-d)S + \sum_{v \in V(S)} v$$

$\rightsquigarrow$ there is unimod φ: $\varphi(S) \subseteq [0, \operatorname{vol} S]^d$

$$\Rightarrow P \cong \varphi(P) \subseteq (-d)\varphi(S) + \sum_{v \in V(S)} \varphi(v)$$
$$\subseteq [0, -d \operatorname{vol} S]^d + \sum_{v \in V(S)} \varphi(v)$$

$\boxtimes$

Corollary (Finiteness Thm)

There are only finitely many isomorphism classes of lattice polytopes of given dimension and volume. $\boxtimes$

5. Counting Lattice Points

16.5

volume vs lattice pts in convex bodies:

B convex, approximate vol by cubes at lattice pts in B:
and refine the lattice:

$$\int_B dx = \lim_{k\to\infty} \frac{1}{k^d} \left| B \cap \frac{1}{k}\mathbb{Z}^k \right| = \lim_{k\to\infty} \frac{1}{k^d} \left| kB \cap \mathbb{Z}^d \right|$$

Def: B convex bounded

$$\mathrm{ehr}_B : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}_{\geq 1}$$
$$k \longmapsto |kB \cap \mathbb{Z}^d|$$

Ehrhart counting function

Ex: • $P = [a,b]$, $a,b \in \mathbb{Z}$ $\rightsquigarrow$ $\mathrm{ehr}_P(k) = k(b-a)+1$

• $P = [0,1]^d$ $\rightsquigarrow$ $\mathrm{ehr}_P(k) = (k+1)^d$

• $P = \Delta_d$ $\rightsquigarrow$ $\mathrm{ehr}_P(k) = \binom{d+k}{d}$ (Exercise)

$\rightsquigarrow$ polynomial of degree d!

Ehrhart's Theorem: $P \subseteq \mathbb{R}^d$ d-dim lattice polytope

The $\mathrm{ehr}_P(k) := |kP \cap \mathbb{Z}^d|$ is a polynomial of deg d.
with leading coeff $\mathrm{vol}\, P$.

The proof requires some preparations:

(1) Look at cone over P:

$$C(P) := \{ (x_0, x_0 x) \mid x_0 \geq 0, x \in P \}$$

recover kP as $C(P) \cap \{x_0 = k\}$

(2) Encode lattice points in a cone by

$$\sum_{a \in C \cap \mathbb{Z}^{d+1}} x^a \quad , \; x^a = x_1^{a_1} \cdots x_d^{a_d}$$

(3) Show that this formal series can be written as a rational function

(4) compute this for simplicial cones

(5) use triangulation for general cones

(6) evaluate at $(k, 1, \dots, 1)$