6. Generating Functions

How should we encode lattice pts in polytopes | cones?

Enumerate?

$P_m := [0,m] \rightsquigarrow P_3 = 0,1,2,3 \quad , \quad P_{1000} = ?$

$\rightsquigarrow$ rewrite $a \in \mathbb{Z}^d$ as $x^a := x_1^{a_1} \cdots x_d^{a_d}$

$$P_3 = \sum_{i=0}^{3} x^i \qquad P_{1000} = \sum_{i=0}^{1000} x^i$$

$\rightsquigarrow$ rewrite as rational function:

$$P_3 : \frac{1-x^4}{1-x} \qquad P_{1000} : \frac{1-x^{1001}}{1-x}$$

now also $P_\infty : \dfrac{1}{1-x}$ for a ray.

Does this generalize?

$$P = [0,\infty) \rightsquigarrow \frac{1}{1-x}$$

$$P = \operatorname{cone}(e_1,e_2) \subseteq \mathbb{R}^2 \rightsquigarrow \frac{1}{1-x_1} \cdot \frac{1}{1-x_2}$$

$$P = \operatorname{cone}\left(\binom{2}{1},\binom{1}{2}\right) \rightsquigarrow (1 + x_1x_2 + x_1^2x_2^2)\,\frac{1}{1-x_1^2x_2}\,\frac{1}{1-x_1x_2^2}$$

$P = \operatorname{cone}(a_1,\ldots,a_d)$ simplicial cone

$$\rightsquigarrow \sum_{a \in \Pi(a_1,\ldots,a_d)} x^a \cdot \prod_{i=1}^{d} \frac{1}{(1-x^{a_i})}$$

But in what sense are

$$\sum_{i\geq 0} x^i \quad \text{and} \quad \frac{1}{1-x} \quad \text{equal ?}$$

can be evaluated for $|x|<1$ defined for all $x \neq 1$

$P = \mathbb{R}$:
$$\sum_{i\in\mathbb{Z}} x^i = \underbrace{\sum_{i<0} x^i}_{\frac{1}{x-1}} + \underbrace{\sum_{i\geq 0} x^i}_{\frac{1}{1-x}}$$

$$\frac{1}{x-1} + \frac{1}{1-x} = 0 \ ?$$

and no common convergence

Formal setting:

left side is <u>formal Laurent series</u> :

$L := \mathbb{k}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ Laurent polynomials

$\hat{L} := \mathbb{k}[[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]]$ L-module of formal Laurent series

<u>Def</u> $G \in \hat{L}$ is <u>summable</u> if there is $g \in L$ s.th. $gG \in L$.

<u>Prop</u>: The set L^{sum} of summable Laurent series is an L-submodule of $\hat{L}$, and the map

$$\varphi : L^{\text{sum}} \longrightarrow R := k(x_1, \ldots, x_d)$$
$$G \longmapsto f/g \quad \text{if } g, f = gG \in L$$

is a homomorphism. ☒

Now: $P = \mathbb{R}$ ⇝ $(1-x)\sum_{i\in\mathbb{Z}} x^i = 0 \Rightarrow \varphi\left(\sum_{i\in\mathbb{Z}} x^i\right) = 0$!

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For a polyhedron P and summable

$G_P(x) = \sum_{a \in P \cap \mathbb{Z}^d} x^a$ define the integer pt generating function

$$g_P(x) = \varphi(G_P(x))$$

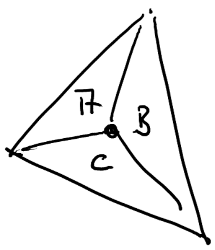
Back to simplicial cones $C = \operatorname{cone}(a_1, \dots, a_d)$ with primitive $a_1, \dots, a_d$ as above: Formally, we have

$$(*) \qquad g_C(x) = \varphi\Big(\sum_{a \in C \cap \mathbb{Z}^d} x^a\Big) = \frac{\sum_{u \in \Pi(a_1, \dots, a_d) \cap \mathbb{Z}^d} x^u}{\prod_{i=1}^d (1 - x^{a_i})}$$

For non-simplicial cones C we triangulate

$G_C(x)$ is just a formal sum of x^a for $x \in C \cap \mathbb{Z}^d$

$\Rightarrow$ we can use inclusion-exclusion to write $G_C(x)$ as a formal sum of $G_F(x)$ for faces F of the triangulation:



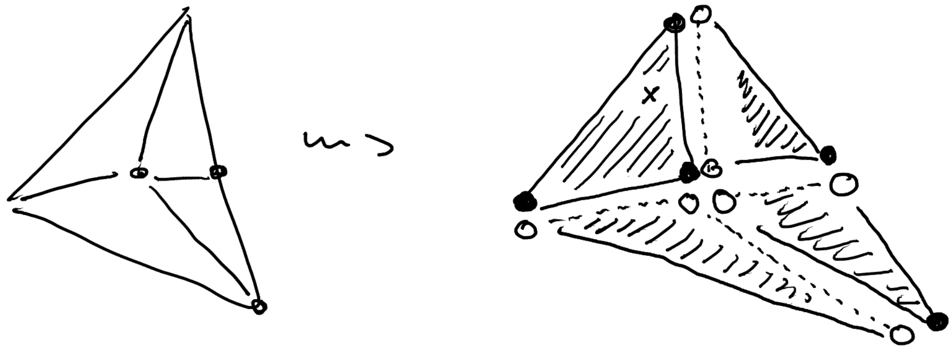
lp in C = lp in A $\cup$ lp in B $\cup$ lp in C
− lp in $A \cap B$ − lp in $B \cap C$ − lp in $A \cap C$
+ lp in $A \cap B \cap C$

This carries over to RHS as φ is a homomorphism!

<u>Coro</u> C polyhedral cone. Then $G_C(x)$ is summable and $g_C(x)$ can be written as a linear combination of rational functions of the form $(*)$ $\boxtimes$

Exercises: We can subdivide a polytope | cone C into "half open" simplices | simplicial cones

$\rightsquigarrow$ max simplices/cones in this subdivision are disjoint and cover C



"take faces x cannot see"

$\rightsquigarrow$ we can generalize our formula for $g_C(x)$ for simplicial cones to half-open cones

$\Rightarrow$ we can write $g_C(x)$ for all cones as sum over the rational functions for half-open cones

$\longrightarrow$ no subtraction of lower-dim contributions no coefficients ± 1 for inclusion-exclusion!

7. Theorem of Ehrhart

P polytope. Then $C(P) := \operatorname{cone}(\{1\} \times P)$

$$= \operatorname{cone}\left((x_0, x_0 x) \mid x_0 \geq 0,\ x \in P \right)$$

$$\rightsquigarrow \mathcal{E}_{C(P)}(t,1,\dots,1) = \sum_{k \geq 0} |kP \cap \mathbb{Z}^d| \, t^k =: 1 + \sum_{k\geq 1} e_P(t) t^k$$

<u>Def</u>: The <u>Ehrhart series</u> of a lattice polytope P is

$$\operatorname{Ehr}_P(t) := 1 + \sum_{k \geq 1} e_P(t) t^k \quad \in \mathbb{k}[[t]]$$

with rational function $\operatorname{ehr}_P(t) := \varphi(\operatorname{Ehr}_P(t)) \in \mathbb{k}(t)$

<u>Thm A</u>: $f, g : \mathbb{R} \to \mathbb{R}$ with $\varphi\left(\sum_{t \geq 0} f(t) z^t\right) = \dfrac{g(z)}{(1-z)^{d+1}}$

Then: f is a polynomial of degree $\leq d$

iff g ———— " ————

and in this case: $f(t) = \sum_{i=0}^{d} g_i \binom{t+d-i}{d}$

and $\deg f = d \iff g(1) \neq 0$.

proof: $\left\{ \binom{t+d-j}{d} \mid 0 \leq j \leq d \right\}$ is a basis of $\mathbb{k}[t]_{\leq d}$

"$\Rightarrow$" f polynomial of degree at most d

Then $f(t) = \sum_{j=0}^{d} g_j \binom{t+d-j}{d}$

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Now: $\sum_{t\geq 0}\sum_{j=0}^{d} g_j \binom{t+d-j}{d} z^t$

$$= \sum_{j=0}^{d} g_j \sum_{t\geq 0} \binom{t+d-j}{d} z^t$$

$\Big\ulcorner$ Ex: $\sum_{t\geq 0}\binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}}$

$\Rightarrow \sum_{t\geq 0}\binom{t+d-j}{d} z^t = \frac{z^j}{(1-z)^{d+1}}$ by shifting $\Big\lrcorner$

$$= \sum_{j=0}^{d} g_j \frac{z^j}{(1-z)^{d+1}} \quad \rightsquigarrow \quad g(z) := \sum g_j z^j$$

"$\Leftarrow$" If g is a polynomial of degree at most d, then the coeff of z^t is $\sum_{j=0}^{d} g_j \binom{t+d-j}{d}$

this is a polynomial of degree at most d in t. $\boxed{\xi}$

Prop $S = \mathrm{conv}(a_0, \ldots, a_d)$ d-simplex,
$C(S) = \mathrm{cone}(\,(1, a_i) \mid 0 \leq i \leq d\,)$

Then

$$\mathrm{Ehr}_S(t) = \frac{h^*(t)}{(1-t)^{d+1}}$$

for a polynomial $h^*(t) = \sum h_k^* t^k$ of degree $\leq d$
and $h_k^* = |\,\Pi(\,(1,a_i) \mid 0 \leq i \leq d) \cap \mathbb{Z}^{d+1} \cap \{x_0 = k\}\,|$

(and therefore $h_0^* = 1$, $h^*(1) \neq 0$)