8.1

proof

$$G_{C(S)}(x) = \frac{\sum_{u \in \Pi((1,a_i)) \cap \mathbb{Z}^{d+1}} x^u}{\prod (1 - x_0 x^{a_i})}$$

substitute $(x_0, x) \to (t, \mathbb{1})$

$$\Rightarrow \mathrm{Ehr}_S(t) = \frac{h^*(t)}{(1-t)^{d+1}}$$

for
$$h^*(t) = \sum_{u \in \Pi(C(S)) \cap \mathbb{Z}^{d+1}} t^{u_0}$$

$\leadsto$ $u_0 \leq d$

$u_0 = 0$ iff $u = 0$. $\boxtimes$

Now the RHS of the equation in this prop has the form of the RHS in the previous theorem. So in

$$\mathrm{Ehr}_S(t) = \sum_{k \geq 0} e_S(k) t^k$$

$e_S(k)$ is a polynomial of degree d with leading coefficient

$$\frac{1}{d!} h^*(1) = \frac{1}{d!} \left| \Pi((1, a_i) \mid 0 \leq i \leq d) \cap \mathbb{Z}^{d+1} \right|$$

$$= \mathrm{vol}\, S$$

For a general polytope we can now triangulate and apply this observation to each face of the triangulation. Using inclusion-exclusion we obtain:

Thm (Ehrhart's Theorem)

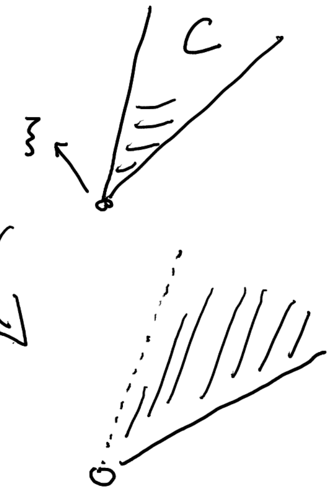
P lattice polytope. Then $e_P(t) := |tP \cap \mathbb{Z}^d|$ extends to a polynomial $e_P(t)$ of degree d and leading coefficient $\operatorname{vol} P$. $\boxtimes$

8. Stanley's Theorems

$C \subseteq \mathbb{R}^d$ d-dim cone, $\xi \in \mathbb{R}^d$ generic (:⇔ not in any facet hyperplane)

we define the half open cone

$$C^\xi := \{ y \in C \mid y + \varepsilon \xi \in C \ \forall \varepsilon \text{ small enough} \}$$



In the exercises:

(1) $\xi \in C$ generic, then

$$C^\xi = C \quad \text{and} \quad C^{-\xi} = \operatorname{int} C$$

(2) T triangulation, T_d max cones of T

ξ generic wrt all cones in T_d, then

$$C = \bigsqcup_{D \in T_d} D^\xi$$

(3) For the simplicial cones D:

$$D^\xi = \left\{ \sum \mu_i v_i \;\middle|\; \begin{array}{ll} \mu_i \geq 0 & : i \in I_+(\xi) \\ \mu_i > 0 & : i \in I_-(\xi) \end{array} \right\}$$

where $\xi = \sum \lambda_i v_i$ and

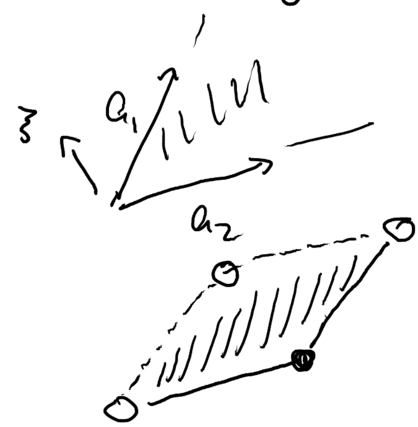
$$I_{+/-}(\xi) := \{ i \mid \lambda_i \underset{<}{\geq} 0 \}$$

Let $D = \operatorname{cone}(a_1, \ldots, a_d)$

for primitive $a_1, \ldots, a_d$

$\rightarrow$ half open parallelepiped

$$\Pi^{\xi}(D) := \Pi^{\xi}(a_1, \ldots, a_d) = \left\{ \sum \mu_i a_i \;\middle|\; \begin{array}{l} 0 \le \mu_i < 1 : i \in I_+(\xi) \\ 0 < \mu_i \le 1 : i \in I_-(\xi) \end{array} \right\}$$



Note:

- $\Pi^{\xi}(D) \subseteq D^{\xi}$
- $\xi \in D$ generic: $\Pi^{\xi}(D) = \Pi(D)$
- in the same way as for $\Pi(D)$
  - any $w \in \mathbb{R}^d$ has a unique rep:

    $w = u + v$ for $u \in \Pi^{\xi}(D)$, $v \in \Lambda(a_1, \ldots, a_d)$
  - the translates $v + \Pi^{\xi}(D)$ for $v \in \Lambda(a_1, \ldots, a_d)$ tile the space

$\Rightarrow$ In the same way as for usual cones we obtain for half open cones that

$$g_{D^{\xi}}(x) = \frac{\prod\limits_{u \in \Pi^{\xi}(V) \cap \mathbb{Z}^d} x^u}{\prod (1 - x^{a_i})}$$

Prop C cone, T triangulation, ξ generic
Then

$$G_C(x) = \sum_{D \in T_d} G_{D^\xi}(x) \quad \text{and} \quad g_C(x) = \sum_{D \in T_d} g_{D^\xi}(x)$$

$\boxtimes$

Note: No inclusion-exclusion involved, so no subtraction on RHS!

Now: P lattice polytope with triang T'
$C(P)$ cone over P with corresponding triang T
$\xi \in C(P)$ generic

We evaluate at $(x_0, x) = (t, \mathbb{1})$:

$$g_{C(P)}(t) = \sum_{D \in T^\xi} \frac{\sum_{(u_0,u) \in \Pi^\xi(D) \cap \mathbb{Z}^{d+1}} t^{u_0}}{(1-t)^{d+1}}$$

Let $h^*_{k,D^\xi} := \left| \{ (u_0,u) \in \Pi^\xi(D) \cap \mathbb{Z}^{d+1} \mid u_0 = k \} \right|$

Then: $h^*_{k,D^\xi} \geq 0 \quad \forall k,$

$h^*_{k,D^\xi} = 0 \quad$ for $k \geq d+1$

$$h^*_{0,D^\xi} = \begin{cases} 1 & : \xi \in \operatorname{int} D \\ 0 & \text{else} \end{cases}$$

Let $h_k^* := \sum_{D \in T_d} h^*_{k,D^\xi}$

Then $h_k^* \geq 0 \ \forall k$

$h_k^* = 0$ for $k \geq d+1$

$h_0^* = 1$

So

$$g_{C(P)}(t) = \sum_{D \in T_d} \frac{\sum_{k=0}^{d} h^*_{k,D^\xi} t^k}{(1-t)^{d+1}}$$

$$= \sum_{k=0}^{d+1} \frac{\sum_{k=0}^{d} h_k^* t^k}{(1-t)^{d+1}}$$

$g_{C(P)}(t) = \mathrm{ehr}_P(t)$, so:

Thm (<u>Stanley's</u> <u>Non-Negativity</u> <u>Theorem</u>)

$P \subseteq \mathbb{R}^d$ d-dim lattice polytope. Then

$$\mathrm{ehr}_P(t) = \frac{\sum_{k=0}^{d} h_k^* t^k}{(1-t)^{d+1}}$$

for a polynomial $h^* : \mathbb{R} \to \mathbb{R}$ of degree at most d and $h_k^* \geq 0 \ \forall k$, $h_0^* = 1$ ⊠

Def: The polynomial h^* is the h^*-polynomial of P.

The <u>degree</u> of P is $\deg P = \deg h^*$.

8.7

Using Thm 7 for last time we get

$$e_P(t) = h_0^*\binom{t+d}{d} + h_1^*\binom{t+d-1}{d} + \dots + h_d^*\binom{t}{d}$$

This implies:

- the constant coefficient of e_P is 1
- $h^*(1) = d!\,\mathrm{vol}\,P = \mathrm{nvol}\,P$
- $h_1^* = |P \cap \mathbb{Z}^d| - d - 1$