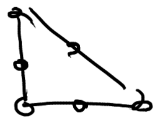
Examples:

- $\Delta_d$ : $\operatorname{vol} \Delta_d = 1 \iff \sum h_i^* = 1$
  $\iff h_0^* = 1, \ h_i^* = 0 \ \forall i \geq 1$

  $P$ polytope with $h^*(t) = 1 \iff P = \Delta_d$

- $P = 2\Delta_2 =$  $h_1^* = 1 \quad h_2^* = 3$
  $\operatorname{vol} P = 4 \implies h_2^* = 0$

  $\implies h^*(t) = 1 + 3t$

<u>Thm</u> (Stanley <u>Monotonicity</u>)

$P, Q$ lattice polytopes, $\dim P = d$, $Q \subseteq P$, with $h^*$-polys $h_P^*, h_Q^*$

Then $h_{Q,i}^* \leq h_{P,i}^*$ for $0 \leq i \leq d$

In particular: $\deg Q \leq \deg P$, $\deg F \leq \deg P$ for all faces.

proof: Ex $\blacksquare$

# 8. Reciprocity



$\rightarrow$ want to count pts in the interior of a lattice polytope $P$

$P = [a,b]$ , $a,b \in \mathbb{Z}$

$$\rightsquigarrow \quad |k\mathring{P} \cap \mathbb{Z}| = k(b-a) - 1$$

$$= -\big( (-k)(b-a) + 1 \big)$$

$$= e_P(k)$$

$P = [0,1]^d$ :

$$|k\mathring{P} \cap \mathbb{Z}^d| = (k-1)^d = (-1)^d \big( (-k)+1 \big)^d$$
$$= (-1)^d \, e_P(-k)$$

$P$ any polygon, then

$$a = i + \frac{b}{2} - 1 \quad \rightsquigarrow$$

$$|k\mathring{P} \cap \mathbb{Z}^2| = i(k) = ak^2 - \frac{b}{2}k + 1$$

$$|kP \cap \mathbb{Z}^2| = i(k) + b(k) = ak^2 + \frac{b}{2}k + 1$$

9.3

Let $a_1, \dots, a_d \in \mathbb{Z}^d$ lin indep, primitive

$D = \mathrm{cone}(a_1, \dots, a_d)$, $m := \sum a_i$

We have a bijection $I_{+/-}(\xi) \longleftrightarrow I_{-/+}(\xi)$

So the map:

$$\begin{aligned}
\alpha : \Pi^{\xi}(D) \cap \mathbb{Z}^d &\longrightarrow \Pi^{-\xi}(D) \cap \mathbb{Z}^d \\
u &\longmapsto m - u \\
\sum \lambda_i a_i &\longrightarrow \sum (1-\lambda_i) a_i
\end{aligned}$$

is well-defined and a bijection.

$$\Rightarrow \sum_{a \in \Pi^{\xi}(D) \cap \mathbb{Z}^d} x^a = \sum_{a \in \Pi^{-\xi}(D) \cap \mathbb{Z}^d} x^{m-a}$$

$$= x^m \sum_{a \in \Pi^{-\xi}(D) \cap \mathbb{Z}^d} x^{-a}$$

This is the numerator polynomial in the generating function of a half open cone

$C$ cone with triangulation $T$; $\xi$ generic:

Then:

$$g_C(x) = \sum_{D \in T_d} g_{D^\xi}(x)$$

$$= \sum_{D \in T_d} \frac{\sum_{a \in \Pi^\xi(D) \cap \mathbb{Z}^d} x^a}{\prod (1 - x^{a_i})}$$

$$= \sum_{D \in T_d} \frac{x^{\mathbb{1}} \sum_{a \in \Pi^{-\xi}(D) \cap \mathbb{Z}^d} x^{-a}}{\prod (1 - x^{a_i})}$$

$$= (-1)^d \sum \frac{\sum_{a \in \Pi^{-\xi}(D) \cap \mathbb{Z}^d} x^{-a}}{\prod (1 - x^{-a})}$$

$$= (-1)^d \sum g_{D^{-\xi}}(x^{-1}) = (-1)^d g_{\operatorname{int} C}(x^{-1})$$

<u>Thm (Stanley's Reciprocity Thm)</u>

For a cone $C$:

$$g_C(x) = (-1)^d g_{\operatorname{int} C}(x^{-1}) \qquad (*)$$

$\boxtimes$

$\to$ Apply this to cones over polytopes

look at both sides separately:

<u>RHS:</u>

$$\varphi\left(\sum |t\mathring{P} \cap \mathbb{Z}^d| t^t\right) = g_{\operatorname{int} C(P)}(t, \mathbb{1})$$

We need

**Lemma:** $\varphi\left(\sum_{k\in\mathbb{Z}} f(k)t^k\right) = 0$

Using this:

$$\varphi\left(\sum_{k\geq 1} e_P(-k)t^k\right) = \varphi\left(\sum_{k\leq -1} e_P(k)t^{-k}\right)$$

$$= \varphi\left(-\sum_{k\geq 0} e_P(k)\, t^{-k}\right)$$

$$= -\,\mathrm{ehr}_P(t^{-1})$$

$$= -\,g_{C(P)}(t^{-1}, \mathbb{1})$$

$$= (-1)^d\, g_{\mathrm{int}\, C}(t, \mathbb{1})$$

$$= (-1)^d\, \varphi\left(\sum |k\mathring{P} \cap \mathbb{Z}^d|\, t^k\right)$$

$$\Rightarrow \quad (-1)^d \sum |k\mathring{P} \cap \mathbb{Z}^d|\, t^k = \sum |kP \cap \mathbb{Z}^d|\, t^k$$

(use: $\varphi$ injective if all exponents in some half-space)

Comparing coefficients implies

**Thm (Ehrhart-MacDonald-Reciprocity)**
$P$ $d$-dim lattice polytope. Then

$$|k\mathring{P} \cap \mathbb{Z}^d| = (-1)^d e_P(-k) \quad \text{for } k \geq 1$$

$\boxtimes$

Comparing coefficients gives

**Thm (Ehrhart-MacDonald-Reciprocity)**

For a $d$-dimensional lattice polytope:

$$|k\mathring{P} \cap \mathbb{Z}^d| = (-1)^d e_P(-k) \quad \text{for } k \geq 1$$

Prove Lemma:

Suffices to prove this for basis $f_m(k) := \binom{m+k}{m}$

We know $\varphi\left(\sum_{k\geq 0} f_m(k)\, t^k\right) = \dfrac{1}{(1+t)^{m+1}}$

Other half:

$$\sum_{k\leq -1} \binom{k+m}{m} t^k = \sum_{k\geq 1} \binom{-k+m}{m} t^{-k}$$

$$= \sum_{k\geq 1} (-1)^m \binom{k-1}{m} t^{-k}$$

$$= t^{-(m+1)} \sum_{k\geq 0} (-1)^m \binom{k+m}{k} t^{-k}$$

so $\varphi(\text{— " —}) = (-1)^m \dfrac{t^{-m+1}}{(1-\frac{1}{t})^{m+1}}$

$$= -\frac{1}{(1-t)^{m+1}}$$

9.7

$P$ $d$-dimensional lattice polytope.

The codegree of $P$ is

$$\operatorname{codeg} P := d+1-\deg P$$

Prop: Let $r := \operatorname{codeg} P$

Then $r = \min_k \left( |k\mathring{P} \cap \mathbb{Z}^d| \neq 0 \right)$

and $|r\mathring{P} \cap \mathbb{Z}^d| = h^*_r$

proof: Ex $\boxed{}$

(1) second proof that $h^*_{\Delta_d}(t) = 1$: first interior lp is in $(d+1)\Delta_d$

(2) $\deg P = d \iff \operatorname{int} P \cap \mathbb{Z}^d \neq \emptyset$

(3) If $P$ has vertices $v_1, v_2$ not on a common face, then

$v_1 + v_2 \in \operatorname{int}(2P)$

$\Rightarrow \operatorname{codeg} P \leq 2 \Rightarrow \deg P \geq d-1$