9. Brion's Theorem

P lattice polytope:

For a face F:

$$T_F P := \{ v \in \mathbb{R}^d \mid \exists\, w \in F, \varepsilon > 0 : w + \varepsilon(w - v) \in P \}$$

tangent cone of F

Note: For $u \in \operatorname{relint} F$:

$$(T_F P - u)^\vee = n_F \quad \text{normal cone of } F \text{ in } P.$$

Thm (Brianchon-Gram-Thm)

$$G_P(t) = \sum_{\emptyset \neq F \text{ face of } P} (-1)^{\dim F} G_{T_F P}(t)$$

We say that a face F is visible from b if for some $u \in \operatorname{relint} F$

$$P \cap \operatorname{conv}(u, b) = \{u\}$$

(visibility is indep. of the choice of u)

If $b \notin T_F P$, then we can separate with a functional a:

$$\langle a, b\rangle > \max \langle a, T_F P\rangle = \max \langle a, P\rangle$$

but then for $0 < \lambda \leq 1$

$$\langle a, \lambda b + (1-\lambda) u\rangle > \max \langle a, T_F P\rangle$$

$$\Rightarrow \operatorname{conv}(b, u) \cap P = \{u\}$$

$$\Rightarrow F \text{ visible from } b$$

Conversely, let $b \in T_F P$

$\rightarrow$ there is $w \in F$: $\quad w + \varepsilon(b - w) \in P$

if $w \in \operatorname{relint} F$: F not visible from P

Otherwise there is $u \in \operatorname{relint} F, \varepsilon > 0$

$$w' := \varepsilon u + (1-\varepsilon) w \in \operatorname{relint} F,$$
$$v' := w + \varepsilon(b - w) \in P$$

Then $\frac{\varepsilon}{1+\varepsilon} b + \frac{1}{1+\varepsilon} w' = \frac{1}{1+\varepsilon}(v' - (1-\varepsilon) w) + \frac{1}{1+\varepsilon}(\varepsilon u + (1-\varepsilon) w)$

$$= \frac{1}{1+\varepsilon} v' + \frac{\varepsilon}{1+\varepsilon} u \in P$$

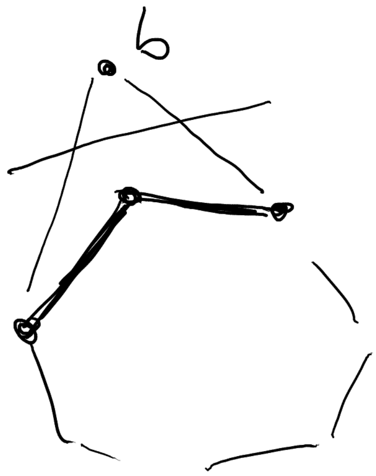
$\Rightarrow F$ not visible from P.

$\Rightarrow F$ visible from $b \iff b \notin T_F P$

The set of visible faces is a subcomplex S of the boundary of P

S is isomorphic to a subdivision of a polytope:

- cone over S with apex b
- intersect with hyperplane and take induced subdivision



Now we can prove the thm:

- think of Laurent series as infinite series where most coeffs are 0.

We compare coefficients of x^m:

(1) $m \in P$

$$\Rightarrow m \in T_F P \quad \forall \quad \emptyset \neq F \text{ face of } P$$

$$\Rightarrow \text{RHS is } \sum_{\emptyset \neq F \text{ face}} (-1)^{\dim F} = 1$$

by Euler

(2) $m \notin P$

$$\Rightarrow \text{RHS}: \sum_{\emptyset \neq F \text{ face}} (-1)^{\dim F} - \sum_{\substack{\emptyset \neq F \text{ visible} \\ \text{from } m}} (-1)^{\dim F}$$

polytope subdivision

$$= 1 \qquad - 1 \qquad = 0 \qquad \boxtimes$$

We observe that most $T_F P$ are not pointed, i.e. contain a line:

$$T_F P \text{ pointed} \iff F \text{ vertex!}$$

But if a cone C contains a line $\mathbb{R}v \subseteq C$, then

$$x^v G_C(x) = G_C(x)$$

$$\Rightarrow \varphi(G_C(x)) = \varphi(x^v G_C(x)) = \varphi(x^v)\varphi(G_C(x))$$

$$\Rightarrow \varphi(G_C(x)) = 0 \;!$$

So if we apply φ to both sides of the equation in the Brianchon-Gram-Thm, we get:

Thm (Brion)

$$g_P(x) = \sum_{v \in V(P)} g_{T_v P}(x)$$

10. Minkowski's Theorems

10.5

→ relates volume and lattice

→ gives estimate for length of shortest vectors

recall: $K \subseteq \mathbb{R}^d$ centrally symmetric $\Leftrightarrow$ $x \in K \Rightarrow -x \in K$

Thm (van der Corput)

K convex, centrally symmetric. Then

$$\mathrm{vol}\, K \leq 2^d \, |K \cap \Lambda| \det \Lambda$$

The inequality is strict if K is compact

→ we need a Lemma first for the proof

→ immediately implies Minkowski's First Theorem:

Thm: K convex, centrally symmetric, $\mathrm{vol}\, K > 2^d \det \Lambda$

Then there is $0 \neq a \in K \cap \Lambda$

For compact K it suffices to assume $K \geq 2^d \det \Lambda$