11. Packings and Coverings

here: $B_r(z) := \{x \in \mathbb{R}^d \mid \|x - z\| < r\}$ open ball

Def: $\rho(\Lambda) := \sup_{r \geq 0} (B_r(x) \cap B_r(y) = \emptyset \ \forall x, y \in \Lambda)$

packing radius

$$\mu(\Lambda) = \max_{x \in \mathbb{R}^d} \operatorname{dist}(x, \Lambda)$$

covering radius

Prop: Λ lattice, a shortest non-zero vector. Then

$$\tfrac{1}{2}\|a\| = \rho(\Lambda)$$ ☒

Recall: Dual lattice $\Lambda^* := \{\alpha : \mathbb{R}^d \to \mathbb{R} \mid \alpha(x) \in \mathbb{Z} \ \forall x \in \Lambda\}$

$\longrightarrow$ lattice in $(\mathbb{R}^d)^*$

Prop:

$$\rho(\Lambda)\,\rho(\Lambda^*) \leq d/4$$

proof: $\rho(\Lambda) \leq \frac{1}{2}\sqrt{d}\,(\det \Lambda)^{1/d}$ $\quad \rho(\Lambda^*) \leq \frac{1}{2}\sqrt{d}\,(\det \Lambda^*)^{1/d}$

and $\det \Lambda \cdot \det \Lambda^* = 1$

☒

Prop: Λ lattice with successive minima $\lambda_1, \dots, \lambda_d$.
Then
$$\mu(\Lambda) \geq \frac{1}{2}\lambda_d \geq \frac{1}{2}\lambda_i \quad \forall i$$

proof: Pick linearly independent vectors v_i sth $\|v_i\| = \lambda_i$

Let $u := \frac{1}{2} v_d$

Assume there is $w \in \Lambda$ sth $d(u,w) < \frac{1}{2}\lambda_d$

$\Rightarrow$
$$\|w\| \leq \|u\| + d(u,w) < \lambda_d = \|v_d\|$$

$\Rightarrow$ By choice of v_d, $w \in \mathrm{lin}(v_1, \dots, v_{d-1})$

$\Rightarrow$ $2w - v_d \notin \mathrm{lin}(v_1, \dots, v_{d-1})$

and $\|2w - v_d\| = \|2(w-u)\| < \lambda_d = \|v_d\|$

This is a contradiction to the choice of v_d

$\Rightarrow$ $\mathrm{dist}(u, \Lambda) = \mathrm{dist}(u, 0) = \frac{1}{2}\lambda_d$

$\Rightarrow$ $\mu(\Lambda) \geq \frac{1}{2}\lambda_d$ $\qquad\blacksquare$

12 Flatness

⇝ convex bodies without interior pts seem to be flat

<u>Def</u>: Λ lattice in $\mathbb{R}^d$ with dual lattice $\Lambda^* \subseteq (\mathbb{R}^d)^*$

For a convex body K and non-zero $a \in \Lambda^*$:

$$\mathrm{width}(K,a) := \max_{x\in K}(a(x)) - \min_{x\in K}(a(x))$$

and

$$\mathrm{width}_\Lambda(K) := \inf(\mathrm{width}(K,a),\ a\in\Lambda^*\setminus\{0\})$$

Note: K full-dim : inf is min
lower dim : width is 0.

We want to prove:

Thm K convex body with $\operatorname{int} K \cap \Lambda = \emptyset$. Then

$$\mathrm{width}_\Lambda(K) \le d^{5/2}$$

We prove this in 3 steps:
for balls, for ellipsoids and then for general K

This requires: • Approximating K by ellipsoids
• Estimating $\rho(\Lambda^*)\cdot\mu(\Lambda)$

So let B be a ball of radius r s.th.

$B \cap \Lambda = \emptyset$ and $v \in \Lambda^*$ a shortest non-zero lattice vector.

Then $\rho(\Lambda^*) = \frac{1}{2}\|v\|$ and $r < \mu(\Lambda)$

$$\Rightarrow \quad \mathrm{width}_v(B) = 2r\|v\| = 4\mu(\Lambda)\rho(\Lambda^*)$$

So we need to bound $\mu(\Lambda)\rho(\Lambda^*)$

Prop

$$1 \leq 4\mu(\Lambda)\rho(\Lambda^*) \leq d^{3/2}$$

proof: We prove the upper bound. The lower bound is left as an Ex.

$\rightarrow$ induction over d

(1) $d = 1 \rightsquigarrow \Lambda = \lambda\mathbb{Z}$ for some λ

$\Lambda^* = \lambda^{-1}\mathbb{Z}$

$\Rightarrow \mu(\Lambda) = \lambda/2 \ , \ \rho(\Lambda^*) = \lambda^{-1}/2$

$\Rightarrow 4\mu(\Lambda)\rho(\Lambda^*) = 1$ ✓

(2) $d > 1$: Let v be shortest in Λ

and $L = v^\perp$ orth. complement with

$\pi : \mathbb{R}^d \rightarrow L$ projection

Let $\Gamma := \pi(\Lambda)$.

$\to$ Γ is a lattice in L, $\Gamma^* \subseteq \Lambda^*$ and

$$\rho(\Gamma^*) \geq \rho(\Lambda^*)$$

Now: bound $\mu(\Lambda)$

Let $x \in \mathbb{R}^d$, $y = \pi(x)$, $u \in \Gamma$ closest pt to y.

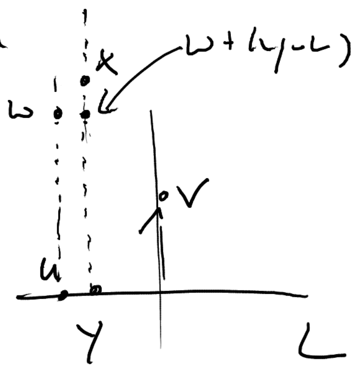
$\Rightarrow$ $\|u - y\| \leq \mu(\Gamma)$

Consider the line $\pi^{-1}(u)$.

$\to$ any two neighboring lattice pts on this line have distance $\|v\|$

$\Rightarrow$ we can pick $w \in \Lambda \cap \pi^{-1}(u)$ st.

$$d(x, w + (y-u)) \leq \tfrac{1}{2}\|v\|$$



Look at the right triangle

x, w, $w+(y-u)$

with a right angle at $w+(y-u)$. We get

$$\|x - w\|^2 = \|x - (w + (y-u))\|^2 + \|y-u\|^2$$

$$\Rightarrow \mu(\Lambda)^2 \leq \|x-w\|^2 \leq \mu(\Gamma)^2 + \tfrac{1}{4}\|v\| = \mu(\Gamma)^2 + \rho(\Lambda)^2$$

$$\Rightarrow \mu(\Lambda)^2 \rho(\Lambda^*)^2 \leq \mu(\Gamma^2)\rho(\Lambda^*)^2 + \rho(\Lambda)^2 \rho(\Lambda^*)^2$$
$$\leq \mu(\Gamma)^2 \rho(\Gamma^*)^2 + \rho(\Lambda)^2 \rho(\Lambda^*)^2$$
$$\leq (d-1)^3 + \frac{1}{16} d^2 \quad < d^3$$

$\boxtimes$

With this proposition we can pick up our consideration about flatness of balls.
We have already seen that for a shortest non-zero $v \in \Lambda^*$

$$\text{width}_v B \leq 4\,\mu(\Lambda)\,\rho(\Lambda^*)$$

With the bound from the proposition we get

$$\text{width}_v B \leq d^{3/2}$$