We want to prove

Thm: K convex body, $K \cap \Lambda = \emptyset$. Then

$$\text{width}_\Lambda K \le d^{5/2}$$

We have already seen that

(1) $\quad 1 \le \rho(\Lambda^*)\,\mu(\Lambda) \le d^{3/2}$

(2) For a shortest $a \in \Lambda^* \setminus \{0\}$

$$\text{width}(K, a) \le d^{3/2}$$

Observe that this bound does not depend on the lattice.

$\Rightarrow$ using an affine linear transformation we obtain the same bound for any ellipsoid E:

Let E be an ellipsoid. Then there is a map

$$\alpha : x \longmapsto Tx + a$$

s.th.

$$\alpha(E) = TE + a \quad \text{is a ball}$$

T is an invertible map, so

$$\Lambda' := T\Lambda \quad \text{is a lattice.}$$

B.2

Now let $v' \in \Lambda'$ be a shortest non-zero vector.

Then
$$v := T^{-1} v' \in \Lambda \setminus \{0\}$$

Hence, for a shortest non-zero $w \in \Lambda$ we get

$$\mathrm{width}_w(E) \le \mathrm{width}_v(E) \le \mathrm{width}_{v'} B \le 6 d^{3/2}$$

Prop: Λ lattice, E ellipsoid with $E \cap \Lambda = \emptyset$

Then $\mathrm{width}_\Lambda(E) \le d^{3/2}$ $\boxed{\Xi}$

Now we need to get from ellipsoids to general convex bodies

<u>Excursion to convex geometry</u>

K convex body (compact convex non-empty set)

Let
$$\eta := \sup(\mathrm{vol}\, E \mid E \subseteq K \text{ ellipsoid})$$

<u>Thm</u>: The supremum is attained by a unique ellipsoid $E \subseteq K$.

Def: This ellipsoid is the <u>Löwner-John</u> ellipsoid.

proof (sketch)

B unit ball, $S := \{ (T, a) : TB + a \subseteq K \}$

T linear, $\in \mathbb{R}^d$

B3

any ellipsoid $E \subseteq K$ is of this form and

$$\operatorname{vol} E = |\det T| \operatorname{vol} B.$$

K is compact $\Rightarrow$ there is r s.th. $\|x\| \leq r \;\forall x \in K$

$\Rightarrow \|a\| \leq r$ and $\|Tx\| \geq r \;\; \forall x \in B$
$(T,a) \in S$

$\Rightarrow S$ is a closed bounded subset of $Gl_d \times \mathbb{R}^d$

$\Rightarrow$ the map $(T,a) \mapsto |\det T|$

attains its maximum (T_0, a_0) in S

Now K is non-empty, so $\det T_0 > 0$
and

$T_0 B + a_0$ is a max. volume ellipsoid.

Uniqueness:

(1) if $T_1 \neq T_2 \rightsquigarrow$ show that $(T_1+T_2)/_2 \, B$
has larger volume and lies in K

(2) if $a_1 \neq a_2$ in $(T,a_1),(T,a_2)$, then
there is an ellipsoid of larger volume contained
in

$$\operatorname{conv}(TB+a_1, TB+a_2) \subseteq K$$

$\boxtimes$

The key theorem is now:

Thm: K convex body, $E \subseteq K$ ellipsoid of max volume.
If the center of E is the origin, then $K \subseteq dE$

(we can always assume that the center is 0 by translation)

proof (sketch)
Using a linear map we can assume that $E = B_d$.

$\rightarrow$ need to show that there is no $x \in K : \|x\| > d$.

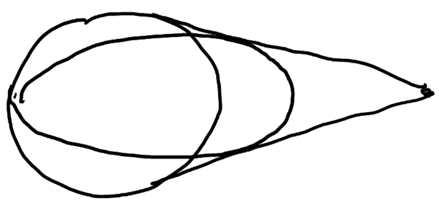
Assume there is such an x.
Using a transformation we can assume $x = me_1$

Let $L := \mathrm{conv}(B_d \cup me_1) \subseteq K$

$\rightsquigarrow$ construct ellipsoid inside L of vol $>$ vol B_d:
Let

$$F_d := \left\{ x \in \mathbb{R}^d \;\middle|\; \frac{1}{a^2}(x_1 - \varepsilon)^2 + \frac{1}{b^2}\sum_{i=2}^{d} x_i^2 \leq 1 \right\}$$

$\rightarrow$ symmetric in last coordinates $\rightsquigarrow$ suffices to look at $d=2$



$\rightarrow$ compute a and b:

$$a = 1+\varepsilon, \qquad b^2 = \frac{(m-\varepsilon)^2 - (1+\varepsilon)^2}{m^2+1}$$

$\Rightarrow \operatorname{vol} F_d = ab^{d-1} \operatorname{vol} B_d$

and
$$ab^{d-1} = (1+\varepsilon)\left(\frac{(m-\varepsilon)^2-(1+\varepsilon)^2}{m^2+1}\right)^{\frac{d-1}{2}}$$

Consider this as function f of ε and compute derivative at $\varepsilon = 0$:
$$f'(0) = \frac{m-d}{m-1} > 0 \quad \text{for } m > d$$

$\boxtimes$

Note: If K is also centrally symmetric, then $K \subseteq \sqrt{d}\, E$!

Back to flatness:

Thm (Flatness Thm)

K convex body with $K \cap \Lambda = \emptyset$. Then
$$\operatorname{width}_\Lambda K \leq d^{5/2}$$

proof: E max volume ellipsoid and v a shortest non-zero lattice vector. Then
$$\operatorname{width}_v E \leq d^{3/2}$$

By using a translation we can assume that the center of E is the origin, so $K \subseteq dE$ and
$$\operatorname{width}_v K \leq d \operatorname{width} E \leq d^{5/2}$$
$\boxtimes$

B.6

Remark: This is not the optimal bound. In fact one can prove

$$\mathrm{width}_1 K \leq O(d^{3/2})$$

It is unknown wether this can be strengthened to $O(d)$

13. Lattice Polytopes with given h^*-Polynomial

Recall h^*-polynomials for Ehrhart Theory:

P lattice polytope

$$\mathrm{Ehr}_P(t) = \sum e_P(k) t^k \qquad \text{Ehrhart series}$$

e_P: Ehrhart polynomial

$$\varphi(\mathrm{Ehr}_P(t)) = \mathrm{ehr}_P(t) = \frac{h^*(t)}{(1-t)^{d+1}} \qquad \text{Ehrhart generating function}$$

$h^*(t)$: h^*-polynomial

(1) $\quad h^*(1) = \sum h^*_k = \operatorname{nvol} P$

(2) $\quad h^*_0 = 1, \; h^*_k \in \mathbb{Z}_{\geq 0}$

(3) $\quad r := \deg h^*, \; s := d+1-r : \; h^*_r = |\operatorname{int} sP \cap \mathbb{Z}^d|$

(4) $\quad h^*_1 = |V(P)| - d - 1$

(5) $\quad Q \subseteq P \;\Rightarrow\; h^*_{Q,k} \leq h^*_{P,k} \quad \forall\, k$

(6) $\quad P = \operatorname{Pyr} Q \;\Rightarrow\; h^*_Q(t) = h^*_P(t)$

(7) For any $V \in \mathbb{R}_{>0}$

$$\left| \{ P \subseteq \mathbb{R}^d \mid P \text{ lattice polytope}, \operatorname{vol} P \leq V \} / \sim \right| < \infty$$

B.8

Let $f(c,s) := c(2s+1)+4s-2$

We want to show:

<u>Thm</u> $P \subseteq \mathbb{R}^d$ lattice polytope, $\deg P = s$, n vertices

if $d > f(n-d-1, s)$
then P is a (multiple) lattice pyramid over a lattice polytope Q of dimension

$$e < f(n-d-1, s)$$