Let $f(c,s) := c(2s+1) + 4s - 2$

We want to show:

<u>Thm</u> $P \subseteq \mathbb{R}^d$ lattice polytope, $\deg P = s$, n vertices

if $d > f(n-d-1, s)$
then P is a (multiple) lattice pyramid over a lattice polytope Q of dimension

$$e < f(n-d-1, s)$$

This implies:

<u>Cor</u> There are (up to deleting lattice pyramids) only finitely many lattice polytopes with a given h^*-polynomial.

proof: $c := h_1^* = n - d - 1$
$s = \deg P$
Thm $\Rightarrow$ we need to consider only lattice polytopes up to dim $f(c,s)$

$h_P^*(1) = \text{nvol } P$

For given volume and dimension there are only finitely many lattice polytopes (up to unimodular transformation). ☐

We prove the theorem in two steps

(1) simplices (2) general polytopes

For a simplex: $c = n - d - 1 = 0$

$\Rightarrow$ need to show that P is a lattice pyramid if $d > 4s - 1$

Let $P = \mathrm{conv}(\bar{v}_0, \dots, \bar{v}_d)$

and $C(P) := \mathrm{cone}(v_0, \dots, v_d)$ for $v_i = (1, \bar{v}_i)$

cone over P

Let $m \in \Pi(v_0, \dots, v_d) \cap \mathbb{Z}$

$$m = \sum \lambda_i v_i \qquad \text{for } 0 \le i < 1$$

$$\mathrm{ht}(m) := \sum \lambda_i \in \mathbb{Z}_{\geq 0}$$

$$\mathrm{supp}\, m := \{ i \mid \lambda_i \neq 0 \}$$

$$\mathrm{supp}\, P := \bigcup_{m \in \Pi \cap \mathbb{Z}^{d+1}} \mathrm{supp}\, m$$

Thm: P is a lattice pyramid with apex v_i for some $i \in \{0, \dots, d\}$ iff $i \notin \mathrm{supp}\, P$.

14.3

Assume that $d \notin \operatorname{supp} P$. Let $L := \operatorname{lin} v_0, \dots, v_{d-1}$

Then $\Pi \cap \mathbb{Z}^{d+1} \subseteq L$

$\leadsto$ pick lattice basis $u_0, \dots, u_{d-1} \in \mathbb{Z}^{d+1}$, $u_d \in \mathbb{Z}^{d+1} \setminus L$

We know that any $u \in \Lambda$ splits uniquely as

$$u = u_\Pi + u_\Lambda \quad \text{for } u_\Pi \in \Pi \text{ and } u_\Lambda \in \Lambda(v_0, \dots, v_d)$$

We get decompositions

$$\underset{\substack{\cap \\ L}}{u_d} = \underset{\substack{\cap \\ \Pi}}{v'} + \underbrace{\underset{\substack{\cap \\ \Lambda(v_0, \dots, v_{d-1})}}{v''} + \mu v_d}_{\in \Lambda(v_0, \dots, v_d)} \qquad v_d = \underset{\substack{\cap \\ L}}{u'} + \lambda u_d$$

$$\Rightarrow u_d = \underbrace{v' + v'' + \mu u'}_{\in L} + \mu\lambda u_d \quad \Rightarrow \mu\lambda = 1 \quad \Rightarrow \mu, \lambda \in \{\pm 1\}$$

Conversely: If v_d has height 1 over L, then

$$\Pi \cap v_d + L = \emptyset$$

and there is no lattice point strictly between L and $L + v_d$ $\boxtimes$

So if $\operatorname{supp} P$ is a proper subset of $\{0, \dots, d\}$, then P is a (multiple) lattice pyramid over

$$B = \operatorname{conv}(v_i \mid i \in \operatorname{supp} P)$$

→ Need to bound the support:

First consider $m = \sum_{i \in \operatorname{supp} m} \lambda_i v_i \in \Pi \cap \mathbb{Z}^{d+1}$

and define $m^* = \sum_{i \in \operatorname{supp} m} (1-\lambda_i) v_i$. Then

$$|\operatorname{supp} m| = \operatorname{ht}(m + m^*) = \operatorname{ht}(m) + \operatorname{ht}(m^*)$$

Now $\operatorname{ht}(m), \operatorname{ht}(m^*) \leq s$ as $\operatorname{ht}(m) = k$ contributes to $h_k^* := |\{m \in \Pi \cap \mathbb{Z}^{d+1} \mid \operatorname{ht}(m) = k\}|$

$$\Rightarrow |\operatorname{supp} m| \leq 2s$$

We choose $m^0, m^1, \dots, m^r \in \Pi \cap \mathbb{Z}^d$ successively s.th.

$$I_j := \operatorname{supp} m^j \setminus \bigcup_{l<j} \operatorname{supp} m^l$$

has maximal size.

Then: $$\operatorname{supp} P = \bigcup I_j$$

Now for fixed k let

$$a := |I_{k-1} \setminus \operatorname{supp} m^k| \quad b = |I_{k-1} \cap \operatorname{supp} m^k| \quad c = |I_k|$$

otherwise we could replace m^{k-1} by m^k

By our choice

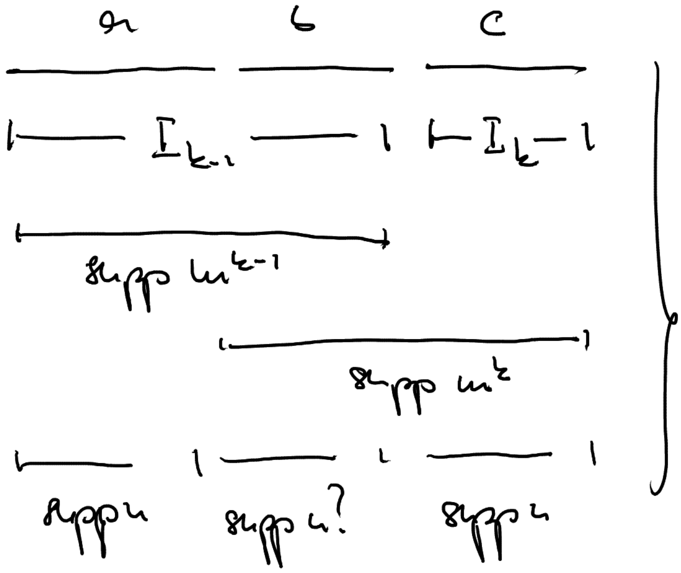
$$a + b = |I_{k-1}|, \quad b + c \leq |I_{k-1}| \Rightarrow c \leq a$$

Now let $$m^k + m^{k-1} = \sum \lambda_i v_i \text{ for some } \lambda_i$$

and $$u := \sum \{\lambda_i\} v_i \in \Pi \cap \mathbb{Z}^{d+1}$$

Now any i contained in only one of the supports of m^{k-1} and m^k is in the support of u, so

$$(\operatorname{supp} m^{k-1} \cup \operatorname{supp} m^k) \setminus (\operatorname{supp} m^{k-1} \cap \operatorname{supp} m^k) \subseteq \operatorname{supp} u$$



$\Rightarrow a + c \leq a + b$ (otherwise choose u for m^{k-1})

Using $c \leq a$ we get

$$|I_k| \leq c \leq \frac{a+b}{2} = \frac{1}{2}|I_{k-1}|$$

This now implies

$$|\operatorname{supp} P| = \sum |I_k| \leq \sum \frac{1}{2^k}|I_0| \leq 2|I_0| \leq 2 \cdot 2s$$
$$= 4s$$

Hence, for a simplex P, P is a lattice pyramid if

$$d \geq 4s - 1$$

14.6

Now we need to extend to general lattice polytopes

$P = \mathrm{conv}(v_0, \ldots, v_n) \quad , n \geq d+1$

$\rightsquigarrow$ there is a circuit A among the vertices of P

$A = A_+ \cup A_- \quad , \quad \sum_{j \in A_+} v_j = \sum_{j \in A_-} v_j = v$

Let $C = \mathrm{conv}\, A$

$\rightsquigarrow \quad v \in |A_+| C$

$\Rightarrow \quad |A_+| \geq \mathrm{codeg}\, C = \dim C + 1 - \deg C$

$= (|A_+| + |A_-| - 2) + 1 - \deg C$

$\Rightarrow \quad |A_-| \leq \deg C + 1$

The same bound holds for $|A_+|$, so

$|A| \leq 2\deg C + 2$

Now we proceed by induction over $c := n - d - 1$

$c = 0 \quad \checkmark \quad$ (P is a simplex)

$c > 0 \quad , \quad d > f(c,s) = c(2s+1) + 4s - 1$

P is not a simplex, so there is a vertex v s.th.

$P' := \mathrm{conv}(V(P) \setminus \{v\})$ is full-dim

Then
- $|V(P')| - \dim P' - 1 < c$
- $\deg P' \leq s$

$\leadsto$ by induction P' is a multiple lattice pyramid over a polytope B' with

$$\dim B' \leq f(|V(P')| - d - 1, \deg P')$$

$$\leq (c-1)(2s+1) + 4s - 1$$

Now $v \in \operatorname{eff} P'$ $\leadsto$ there is a circuit A in P containing v

Let $B := \operatorname{conv}(B' \cup A)$ and

$$\dim B \leq \dim B' + \overbrace{|A| - 1}$$ There is an affine dependence among A!

$$\leq (c-1)(2s+1) + 4s - 1 + \underbrace{|A|}_{\leq 2s+2} - 1 = f(c,s)$$

as $v \in B$, P is a lattice pyramid over B

$\boxtimes$